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MATHEMATICS magazine

MATHEMATICS MAGAZINE

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Editorial Notices:

The introductory paper on Fermat's Last Theorem planned for this issue was crowded forward to the March-April issue by prior commitments.

Two or three libraries requested us to print the indexes seperately. We thought we would do so but found the cost did not justify it. Hence they will continue as before to be bound in the first issue of the next volume. However, we always have quite a few loose ones, printed with the reprints, and will furnish these when requested, as long as they last.

Please note the new rates for reprints. These have been run in the two previous issues and will be effective with the present issue.

IRBY COGHILL NICHOLS

Born in Eudora, Mississippi, November 11, 1882, Irby Coghill Nichols died in Baton Rouge, Louisiana, March 13, 1952.

"Nobody knows like I do what a great life it was."

These words from *Pauline Wright Nichols* in a letter to a friend, stand complete testimony -that of a *wife*- to the *nobility* of the subject of this memorial.

All his life he struggled against physical handicaps, but never with the slightest abatement of early ambitions. Of good things he desired he must attain the *best*.

In 1906 and 1908 he received B.S. and M.S. degrees at the University of Mississippi. Teaching mathematics at Texas A. & M. from 1909 to 1917 he was granted intermission periods for extended graduate work. In 1912 he was awarded the M.S. degree by the University of Illinois. In 1917 came the Ph.D. award from the University of Michigan for his research in the history of the Hindu art of reckoning.

Appointed in 1917 to an Associate Professorship in Louisiana State University, he was made Professor of Applied Mathematics there in 1919- a position which he held until his voluntary retirement in 1947 to give closer attention to his enlarged business and other interests.

Notwithstanding his exceptional power for detailed analysis, his love of the practical together with an instinct to make mathematics an instrument of *service* (He loved people!) naturally led to his assuming a variety of public service roles. Mathematics of investment, of insurance, of probability, and of related fields, were treated by him in various papers before the Louisiana-Mississippi Section of M.A. of A., the Louisiana Academy of Science, or the Louisiana-Mississippi Branch of N.C. of T. of M.

An almost uncanny talent for accuracy led to his holding secretaryships for long periods in the Louisiana-Mississippi Section of M.A. of A. and the Louisiana Academy of Science. Added to these activities were his successive editorial services on the staffs of *Mathematics News Letter, National Mathematics Magazine* and *Mathematics Magazine.*

S. T. Sanders, Professor Emeritus,
La. State University

ON BOUNDS OF POLYNOMIALS IN HYPERSPHERES AND FRECHET-MICHAL DERIVATIVES IN REAL AND COMPLEX NORMED LINEAR SPACES

A. D. Michal

Introduction: One of the main difficulties—by no means the only one—in the development of a theory of analytic functions in normed linear spaces stems from the fact that the modulus of a homogeneous polynomial and that of its polar are not always the same. We give here the theorem that asserts that the modulus of a homogeneous polynomial and its polar are always equal in *complex*[1] Banach spaces. This has many important implications including the great simplifications brought about in the proofs of several fundamental theorems. The situation in *real*[2] Banach spaces is quite different as we shall show. We base our proofs on a few fundamental lemmas in normed linear spaces: one generalizes Bernstein's theorem[3] on the bounds and first derivative of S, a polynomial in the complex plane and two others generalize theorems of the brothers, A. Markoff[4] and V. Markoff[5] on the first and higher derivatives of polynomials of a real variable. These lemmas have an interest in themselves and have many applications not discussed in this paper. To present our results as briefly as possible, we assume familiarity with the Frechet differential calculus and analytic function theory in both real and complex Banach spaces.

Theorems in Real Banach Spaces. The result of A. Markoff states that *if $p_n(x)$ is a polynomial of degree n in a real variable x and if $|p_n(x)| \leq 1$ in the interval $|x| \leq 1$, then the derivative $p_n'(x)$ satisfies the inequality $|p_n'(x)| \leq n^2$ in $|x| \leq 1$.*

The following lemma generalizes Markoff's result.

Lemma 1. Let

$$p(x) = \sum_{i=0}^{n} h_i(x)$$

be a polynomial of degree n on a real Banach space E_1 to a real Banach space E_2, and let E_3 be the usual Banach space of linear (additive and continous) transformations on E_1 to E_2. If $||p(x)|| \leq 1$ in the unit hypersphere $||x|| \leq 1$, then $||p'(x)|| \leq n^2$ in $||x|| \leq 1$, where $p'(x)$ is the Frechet-Michal derivative[6] *of $p(x)$ at $x = x$ with values in E_3.*

Proof. Define $p_n(\lambda)$, a polynomial of degree at most n, on the reals to E_2, by $p_n(\lambda) = p(x + \lambda y)$ for each x, $y \varepsilon E_1$. Let a be an arbitrarily chosen number such that $0 < a < 1$. Hence $||p_n(\lambda)|| \leq 1$ in the interval $|\lambda| \leq 1$ uniformly in x and y for $||x|| \leq 1 - a$ and $||y|| \leq a$.

If $L(x)$ is any linear functional of modulus unity on E_2 to the reals, then from the above inequality

$$|L(p_n(\lambda))| \leq 1 \text{ in } |\lambda| \leq 1$$

and by A. Markoff's inequalities for polynomials of a real variable, we obtain

$$\left\{ \begin{array}{l} |L(p_n'(\lambda))| \leq n^2 \text{ in } |\lambda| \leq 1 \text{ uniformly in } x \text{ and } y \text{ for } ||x|| \\ \leq 1-a \text{ and } ||y|| \leq a. \end{array} \right\} \quad (1)$$

By a well-known theorem in Banach spaces, there exists a linear functional $L(x)$ of modulus unity for each value of $p_n'(\lambda)$ such that $L(p_n'(\lambda)) = ||p_n'(\lambda)||$. But $p_n'(\lambda) = p(x + \lambda y; y)$ for all x, $y \in E_1$ and real λ, where $p(x;y)$ is the Frechet differential of $p(x)$ at $x = x$ with increment y for all x, $y \in E_1$. Hence from (1) we obtain

$$\left\{ \begin{array}{l} ||p(x + \lambda y; y)|| \leq n^2 \text{ in } |\lambda| \leq 1 \text{ uniformly in } x \text{ and } y \text{ for } ||x|| \\ \leq 1-a, \ ||y|| \leq a. \end{array} \right\} \quad (2)$$

Let $z \neq 0$ be any element of E_1. Take $y = \dfrac{az}{||z||}$. Clearly $||y|| = a$ for all $z \neq 0$ in E_1. If we substitute these y's in (2) we obtain

$$\left\{ \begin{array}{l} a||p(x + \lambda \dfrac{az}{||z||}; z)|| \leq n^2||z|| \text{ for } |\lambda| \leq 1, \ ||x|| \leq 1-a \text{ and} \\ \text{all } z \neq 0 \text{ in } E_1. \end{array} \right\} \quad (3)$$

In particular, (3) holds for $a = a_i$ where $\{a_i\}$ is any sequence of positive numbers less than one and converging to one. In the limit we obtain

$||p(\dfrac{\lambda z}{||z||}; z)|| \leq n^2 \, ||z||$ for $|\lambda| \leq 1$ and $z \neq 0$ in E_1, and then clearly

$||p(w;z)|| \leq n^2||z||$ for all w in $||w|| \leq 1$ and all $z \in E_1$. The lemma follows on using the definition of a Frechet-Michal derivative $p'(w)$ of $p(w)$. Q.E.D.

One can obtain the evident bound $n^2(n-1)^2 \cdots (n-(p-1))^2$ for the pth Frechet-Michal derivative of $p(w)$ under the same hypothesis as in Lemma 1. However a better bound can be obtained by proceeding differently.

We assume now that the hypothesis of Lemma 1 is satisfied and proceed as in the proof of Lemma 1 and consider $L(p_n^{(p)}(\lambda))$. There is a result (1892) of W. Markoff that states that *if $p(\lambda)$ is a real polynomial of a real variable λ and if $|p(\lambda)| \leq 1$ in $|\lambda| \leq 1$, then the pth derivative $p^{(p)}(\lambda)$ satisfies the inequality*

$$\left\{ |p^{(p)}(\lambda)| \leq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(p-1)^2)}{1\cdot 3\cdot 5 \cdots (2p-1)} \right\} \quad (4)$$

for all λ in the interval $|\lambda| \leq 1$. On applying this result to $L(p_n^{(p)}(\lambda))$ we obtain the inequality

$$||p(x + \lambda y; \overbrace{y; \cdots; y}^{(p)})|| \leq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(p-1)^2)}{1\cdot 3\cdot 5\cdots(2p-1)}$$

for $|\lambda| \leq 1$ uniformly in x and y for $||x|| \leq 1-a$ and $||y|| \leq a$, where $p(x; \overbrace{y; \cdots; y}^{(p)})$ is the pth successive Frechet differential of $p(x)$ at $x = x$ with p equal increments y. We are thus led to the following lemma.

Lemma 2. *If $p(x)$ is an nth degree polynomial on a real Banach space E_1 to a real Banach space E_2 and if $||p(x)|| \leq 1$ in $||x|| \leq 1$, then the pth successive Frechet differential of $p(x)$ satisfies the modular inequality*

$$||p(x; \overbrace{y; y; \cdots; y}^{(p)})|| \leq \frac{n^2(n^2-1^2)(n^2-2^2)\cdots(n^2-(p-1)^2)}{1\cdot 3\cdot 5\cdot\ldots(2p-1)}||y||^p \qquad (p \geq 1) \quad (5)$$

for all x in $||x|| \leq 1$ and all $y \in E_1$.

Now $p(x; y; y; \cdots; y)$ with p y's is for each $y \in E_1$, a polynomial in x of degree at most $n-p$. Define

$$\left\{ W_{np} = \frac{n^2(n^2-1^2)\ldots(n^2-(p-1)^2)}{1.3\cdot 5\cdot\ldots(2p-1)} \; . \right\} \qquad (6)$$

Now if we consider

$$\frac{p(x; y; y; \ldots; y)}{W_{np}}$$

as a function of x, we can use (5) successively and obtain higher differentials in x of $p(x)$ with different increments. Unfortunately this method does not yield very good bounds. Much better bounds can be obtained by applying (5) to

$$\frac{p(x; y; y; \ldots; y)}{W_{np}}$$

when we consider it as a homogeneous polynomial of degree p in the increment y for each x in $||x|| \leq 1$, and then continue this process successively in the new increment variables.

Let us then define the following homogeneous polynomials:

$$\left\{\begin{aligned} \pi(y) &= \frac{p(x;\ y;\cdots;\ y)}{W_{np}} && \text{with } p\ y\text{'s};\\ \pi^{(i)}(y_i) &= \frac{\pi^{(i-1)}(y_{i-1};\ y_i;\ y_i;\cdots;\ y_i)}{W_{p_{i-1}p_i}} && \text{with } p_i\ y_i\text{'s};\\ \pi^{(1)}(y_1) &= \frac{\pi(y;\ y_1;\ y_1;\cdots;\ y_1)}{W_{pp_1}} && \text{with } p_1\ y_1\text{'s};\\ &(i = 2,\ 3,\cdots,\ r). \end{aligned}\right\} \tag{7}$$

Evidently $\pi^{(i)}(y_i)$ is a homogeneous polynomial of degree p_i in y_i. By Lemma 2

$||\pi(y)|| \le 1$ for $||x|| \le 1$, $||y|| \le 1$, if $||p(x)|| \le 1$ for $||x|| \le 1$. Using Lemma 2 repeatedly, we find $||\pi^{(i)}(y_i)|| \le 1$ for $||x|| \le 1$, $||y_i|| < 1 (i=1,2,\cdots,r)$. If we define $s^{(t)} = (s-1)(s-2)\cdots(s-t+1)$ and expand $||\pi^{*}(y_i)|| \le 1$ we are led to the following result.

Lemma 3. *If $p(x)$ is an nth degree polynomial on a real Banach space E_1 to a real Banach space E_2 and if $||p(x)|| \le 1$ in $||x|| \le 1$, then*

$$||p(x;\ y;\cdots;\ y;\ y_1;\cdots;\ y_1;\ y_2;\cdots;\ y_2;\cdots;\ y_{r-1};\cdots;\ y_{r-1};\ y_r;\cdots;\ y_r)|| \le \frac{W_{np}W_{pp_1}W_{p_1p_2}\cdots W_{p_{r-2}p_{r-1}}W_{p_{r-1}p_r}}{p^{(p_1)}p_1^{(p_2)}\cdots p_{r-2}^{(p_{r-1})}p_{r-1}^{(p_r)}}\,||y||^{p-p_1}||y_1||^{p_1-p_2}\cdots ||y_{r-1}||^{p_{r-1}-p_r}||y_r||^{p_r}$$

$(n \ge p > p_1 > p_2 > \cdots > p_{r-1} > p_r \ge 1,\ r \ge 1,\ p > p_1+p_2+\cdots+p_r)$ for all x in $||x|| \ge 1$ and all $y, y_1, \cdots, y_r \in E_1$ with the understanding that there are $p-p_1$ y's, p_j-p_{j+1} y_j's $(j = 1, 2, \cdots, r-1)$ and p_r y_r's.

The special case $r = p-1$ in Lemma 3 has a particular significance for us. This requires $p \ge 2$, $p_j - p_{j+1} = 1$, $p_1 = p-1$. If we use Lemma 3 for the special case $r = p-1$ and evaluate the constants we finally obtain **Theorem 1**: *If $p(x)$ is an nth degree polynomial on a real Banach space E_1 to a real Banach space E_2 and if $||p(x)|| \le 1$ in $||x|| \le 1$, then the pth successive, Frechet differential of $p(x)$ with increments $y_1, y_2, \cdots y_p$ satisfies the following modular inequality*

$$||p(x;\ y_1;\ y_2;\ y_p)|| \le \frac{2^{\frac{(p-1)(p-2)}{2}}}{(p-1)!\,[1\cdot 3\cdot 5\cdots(2p-5)\ (2p-3)]}\,W_{np}||y_1||\,||y_2||\cdots||y_p||$$

for all x in $||x|| \le 1$ and all $y_1, y_2, \cdots y_p \in E_1$.

We can now consider as established a theorem that follows without difficulty from Theorem 1. It is, in fact, a generalization to polynomials in real Banach spaces of W. Markoff's result on the bounds of

the pth derivative of a real polynomial of a real variable.

Theorem 2 *If $p(x)$ is an nth degree polynomial on a real Banach space E_1 to a real Banach space E_2 and if $||p(x)|| \leq 1$ in $||x|| \leq 1$, then the pth Frechet-Michal derivative $p^{(p)}(x)$ of $p(x)$ satisfies the inequality*

$$||p^{(p)}(x)|| \leq \frac{2^{\frac{(p-1)(p-2)}{2}} W_{np}}{(p-1)!\,[1\cdot3\cdot5\cdots(2p-5)(2p-3)]} \qquad (p \geq 2)$$

in $||x|| \leq 1$.

Going to a still more special case in Lemma 3, we consider $p(x)$ to be a homogeneous polynomial $h_n(x)$ of degree $n \geq 2$ with, say, a polar $W_n(x_1, x_2, \cdots, x_n)$.
Then

$$||W_n(\overbrace{x, \cdots, x}^{n-p}, y_1, y_2, \cdots y_p)|| < \frac{2^{\frac{(p-1)(p-2)}{2}}}{n^{(p)}(p-1)!\,[1\cdot3\cdot5\cdots(2p-5)(2p-3)]} W_{np}$$

$(p \geq 2)$ for all x in $||x|| \leq 1$ and all $y_1, \cdots, y_p \in E_1$.

Hence for $p = n$, we obtain the following theorem after a simplification of the constant.

Theorem 3. *If $h_n(x)$ is an nth degree homogeneous polynomial on a real Banach space E_1 to a real Banach space E_2 with $W_n(x_1, \cdots, x_n)$ as polar, then*

$$||W_n(y_1, y_2, \cdots y_n)|| \leq \frac{2^{\frac{n(n-1)}{2}}}{(n-1)!\,[1\cdot3\cdot5\cdots(2n-3)]}$$

for $||y_i|| \leq 1$ $(i = 1, 2, \cdots, n)$, if $||h_n(x)|| \leq 1$ for $||x|| \leq 1$, $(n \geq 2)$

If $\mu(h_n)$ and $\mu(W_n)$ are the moduli respectively of an arbitrary homogeneous polynomial $h_n(x)$ and its polar $W_n(x_1, \cdots, x_n)$, then it is well known in the Michal-Martin theory that

$$\left\{ 1 \leq \frac{\mu(W_n)}{\mu(h_n)} \leq \frac{n^n}{n!} \right\} \qquad (8)$$

irrespective of whether the spaces are real or complex Banach spaces.

With the aid of Theorem 3 the following result can be proved readily.

Theorem 4. *If $h_n(x)$ is a homogeneous polynomial on a real Banach space E_1, to a real Banach space E_2 with $W_n(x_1, \cdots, x_n)$ as polar, then the moduli $\mu(h_n)$ and $\mu(W_n)$ satisfy the inequalities*

$$\left\{ 1 < \frac{\mu(W_n)}{\mu(h_n)} \le \frac{2^{\frac{n(n-1)}{2}}}{(n-1)!\,[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)]} \right\} \quad (n \ge 2) \tag{9}$$

The question arises whether the inequalities of Theorem 4 are an improvement over the classical Michal-Martin results (8). The following theorem embodies the answer to this question.

Theorem 5. *Under the hypothesis of Theorem 4, the modular inequalities* (9) *are an improvement over the classical inequalities* (8) *for homogeneous polynomials* $h_n(x)$ *of degree* $3 \le n \le 14$ *while for all* $n \ge 15$ *the classical Michal-Martin results* (8) *are still definitely better than* (9).

The proof of Theorem 5 is made to depend in the following arithmetical lemma.

Lemma 4

$$\left\{ 2^{\frac{n(n-1)}{2}} < n^{n-1}[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)] \right\} \tag{10}$$

for all positive integers n *satisfying* $3 \le n \le 14$ *while the reverse inequality*

$$\left\{ 2^{\frac{n(n-1)}{2}} > n^{n-1}[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)] \right\} \tag{11}$$

holds for all positive integers $n \ge 15$.

Proof. On taking logarithms and using Stirling's approximation formula, it is easy to see that the logarithm of the left side of inequality (10) is of order n^2 while that of the right side is of smaller order. Hence for sufficiently large n, we obtain (11) instead of (10). By the evident long but elementary calculations it is found that (10) holds for $3 \le n \le 14$ and that (11) holds for $n = 15$. An induction shows that (11) holds for all $n \ge 15$.

It is evident from Theorem 5 that for questions of convergence of power series in real Banach spaces, the new inequalities (9) are weaker than the old inequalities (8).

Theorems in Complex Banach Spaces. We turn our attention to the simpler situations presented by variables in complex Banach spaces. In the first place we recall S. Bernstein's theorem in the complex plane. *If* $p_n(x)$ *is a polynomial of degree* n *in a complex variable* x, *and if* $|p_n(x)| \le 1$ *in the circle* $|x| \le 1$, *then the derivative* $p_n'(x)$ *satisfies the inequality*

$$|p_n'(x)| \le n \text{ in } |x| \le 1.$$

It is possible to generalize Bernstein's results to complex Banach

spaces. The following lemma can be proved.

Lemma 5. Let $p(x) = \sum_{i=0}^{n} h_i(x)$ be a polynomial of degree n on a complex Banach space E_1 to a complex Banach space E_2, and let E_3 be the complex Banach space of linear (additive, homogeneous of degree one with respect to complex number multipliers, and continuous) transformations on E_1 to E_2. If $||p(x)|| \leq 1$ in the unit hypersphere $||x|| \leq 1$, then

$$||p'(x)|| \leq n \text{ in } ||x|| \leq 1,$$

where $p'(x)$ is the Michal-Frechet derivative of $p(x)$ at $x = x$ with values in E_3.

Proof: We can give the details of proof very briefly by describing the few changes needed in the method used in the proof of Lemma 1. Use Bernstein's result in the place of Markoff's result thus entailing a change from n^2 to n throughout the proof. Of course all concepts used are those of complex Banach spaces and not those of real Banach spaces. Q.E.D.

The following theorems (and others to be discussed elsewhere) are important and have far-reaching consequences. They are obtained from Lemma 5 without difficulty.

Theorem 6. The moduli of a homogeneous polynomial $p_n(x)$ and its polar $W_n(x_1, x_2, \cdots x_n)$ are equal in complex Banach spaces.

This theorem does not hold in general in real Banach spaces—see Theorem 4.

Theorem 7[7]. If $f(x)$ has arguments and values in complex Banach spaces E_1 and E_2 respectively, and if $f(x)$ is Michal-Martin analytic in $||x|| < r$ (so that $f(x) = \sum_{n=0}^{\infty} p_n(x)$ for $||x|| < r$ and $\sum_{n=0}^{\infty} \mu(p_n(x))\lambda^n$ is convergent for $\lambda < r$ with $\mu(p_n(x))$ the modulus of the homogeneous polynomial $p_n(x)$ of degree n), then $f(x)$ has Frechet differentials of all orders that are Michal-Martin analytic in $||x|| < r$ and are obtained by taking the corresponding Frechet differentials of the abstract power series representing $f(x)$ in $||x|| < r$ term by term in $||x|| < r$. These successive Frechet differentials of $f(x)$ are Michal-Martin analytic functions of x in $||x|| < r$ for each value of the increment in E_1.

Theorem 8. Let E_1 and E_2 be complex Banach spaces. If $f(x) = \sum_{n=0}^{\infty} p_n(x)$ has arguments and values in E_1 and E_2 respectively, and is Michal-Martin analytic in $||x|| < r$, then it is analytic in some neighborhood $||x-x_0|| < \bar{r}_{x_0}$ at every point x_0 in $||x|| < r$. In fact $\bar{r}_{x_0} \geq r - ||x_0||$.

Theorem 6 is proved by repeated use of Lemma 5 while Theorem 7 and Theorem 8 are proved by a slight variation of the classical complex variable methods with the aid of the same Lemma 5. Theorem 7 and Theorem 8 are known theorems in the Michal-Martin theory of analytic functions. Although they were used widely in the literature since

1932 their complicated proofs (Lemma 5 was unknown) were never published—although they are given explicitly in Martin's California Institute of Technology Thesis, 1932. Slight variations of these proofs are also given in Volume I of the author's forthcoming monograph: Michal, A.D. Differential Equations in Abstract Spaces with Applications.

References

1. An exposition of complex Banach spaces is given in *Hille, E.*, "Functional Analysis and Semi Groups", New York (1948). For a reasonably complete discussion of the Michal-Martin theory (1931-1932) of analytic functions in complex Banach spaces see a forthcoming monograph by the author: *Michal, A.D.* "Differential Equations in Abstract Spaces with Applications".
2. For real Banach spaces, see *Banach, S.*, "Operations Lineaires", Warsaw (1932). For analytic functions in real Banach spaces see *Michal, A.D.*, loc. cit.
3. *Fekete, M.*, "Uber einen Satz des Herrn Serge Bernstein", Journ, fur die reine und ang. Math., 88-94, vol. 146 (1916).
4. *Markoff, A.*, "On a certain problem of D.F. Mendeleief", Utcheniya Zapiski Imperatorskoi Akademci Nauk (Russia), vol. 62 (1889) pp. 1-24.
5. *Markoff, W.*, "Uber Polynome, die in einen gegebenen Intervalle moglichst wenig von Null abweichen", Math. Annalen, vol. 77 (1916), pp. 213-258 - published earlier in Russian in 1892.
6. Let $f(x)$ be a function on an open set $S \subset E_1$ to E_2 where E_1 and E_2 are Banach spaces (real or complex). If $f(x)$ has a Frechet differential $f(x_0; v)$ at $x = x_0 \varepsilon S$ with increment v, then we can define a bilinear function $f'(x_0)\cdot y$ on E_3 E_1 to E_2, where E_2 is the usual Banach space of linear transformations on E_1 to E_2, by

$$f(x_0; y) = f'(x_0)\cdot v \text{ for } y \varepsilon E_1.$$

Then $f'(x_0)$, by definition, is the Michal-Frechet (or Frechet-Michal) derivative of $f(x)$ at $x = x_0$. Perhaps "differential coefficient" instead of "derivative" would be just as appropriate.
7. The proof of the analogue of Theorem 7 for real Banach spaces is proved in *Michal, A.D.*, Duke Math. Journ. (1946).

California Institute of Technology

THE HARMONY OF THE WORLD

Morris Kline

From Harmony, from heavenly harmony,
This universal frame began:

Dryden

Among the many contributions of mathematics to modern civilization the most valuable are not those which serve the physicist and the engineer but rather those which have fashioned our culture and our intellectual climate. Not enough people are aware of the latter contributions, not even of the role mathematics played in the greatest revolution in the history of human thought - the establishment of the heliocentric theory of planetary motions. It is a fact of history that mathematics forged this theory and was the sole argument for it at the time that it was advanced. No more impressive illustration of the enormous influence mathematics has had on modern culture can be found than that which it exerted through its contributions to the heliocentric doctrine.

The first publication on this theory was Copernicus's *ON THE REVOLUTIONS OF THE HEAVENLY SPHERES* (1543). The title page of this work gave advance notice of the important role mathematics was to play for on this page appeared the legend originally inscribed on the entrance to Plato's academy: "Let no one ignorant of geometry enter here." With this publication the Renaissance gave to the world one of its finest fruits.

Perhaps the enterprising merchants of the Italian towns got more than they bargained for when they aided the revival of Greek culture. They sought merely to promote a freer atmosphere; they reaped a whirlwind. Instead of continuing to dwell and prosper on firm ground, the *TERRA FIRMA* of an immovable earth, they found themselves clinging precariously to a rapidly spinning globe which was speeding about the sun at an inconceivable rate. It was probably sorry recompense to these merchants that the very same theory which shook the earth free also freed the mind of man.

The fertile soil for these new blossoms of the mind was the reviving Italian universities. There Nicolaus Copernicus became imbued with the Greek conviction that nature is a harmonious medley of mathematical laws and there too he learned of the suggestion of the ancient Greek, Aristarchus, that the behavior of the planets might be describable by regarding them as moving about a stationary sun and by introducing daily rotation of the earth. In Copernicus's

mind these two ideas coalesced. Harmony in the universe demanded a heliocentric theory and he became willing to move heaven and earth in order to establish it.

Copernicus was born in Poland. After studying mathematics and science at the University of Cracow, he decided to go to Bologna, where learning was more widespread. There he studied astronomy under the influential teacher Novara, a foremost Pythagorean. It was at Bologna, also, that he came into contact with Greek culture and Greek astronomy.

In the year 1500 Copernicus was appointed canon of the Cathedral of Frauenberg in East Prussia but got leave to continue his studies in Italy. For the next few years, he studied medicine and canon law, securing a doctor's degree in each field, and continued his thinking on astronomical problems. In 1512 he finally assumed the duties of his position at Frauenberg, but spent much time during the remaining thirty one years of his life in a little tower on the wall of the cathedral closely observing the planets with naked eye and making untold measurements with crude home-made instruments. The rest of his spare time he devoted to improving his new theory on the motions of heavenly bodies.

After years of mathematical reflection and observation, Copernicus finally circulated a manuscript describing his theory and his work on it. The reigning pope, Clement VII, approved of the work and requested publication. But Copernicus hesitated. The tenure of office of the Renaissance popes was rather brief and a liberal pope might readily be succeeded by a reactionary one.

Ten years later Copernicus's friend Rheticus persuaded him to allow the publication, which Rheticus then undertook. A Lutheran pastor, Osiander, took it upon himself to add in press an apologetic preface claiming for the work merely the status of a hypothesis which facilitated astronomical calculations. This preface, Osiander believed, would preclude suppression of the book and permit it to make its way in a hostile world.

While lying paralyzed from an apopleptic stroke, Copernicus received a copy of his book. It is unlikely that he was able to read it for he never recovered. He died shortly afterwards, in the year 1543, having contributed to the world in overflowing measure for the seventy years allotted to him.

At the time that Copernicus delved into astronomy the science was practically in the state in which Ptolemy had left it. However, it had become increasingly difficult to include under the Ptolemaic heavens the knowledge and observations of earth and sky accumulated, largely by the Arabians, during the succeeding centuries. By Copernicus's time it was necessary to invoke a total of seventy seven mathematical circles in order to account for the motion of the sun, moon, and five planets under the geocentric scheme created by

Hipparchus and Ptolemy. No wonder that Copernicus grasped at the possibilities in the Greek idea of planetary motion about a stationary sun.

Copernicus took over some other ideas from the Greeks. Because the latter believed that the natural motion of bodies was circular, he, too, used the circle as the basic curve on which to build his explanation of the motions of the heavenly bodies. For a mystic reason similar to that held by the Greeks he also retained the notion that the speed of any heavenly object must be constant. A change in speed, he reasoned, could be caused only by a change in motive power, and since God, the cause of the motion, was constant, the effect could not be otherwise.

Then Copernicus proceeded to do what no Greek had ever attempted: he carried out the mathematical analysis required by the heliocentric hypothesis. His basic idea was still that used by Hipparchus and Ptolemy, that is, epicycles. In accordance with this scheme the motion of a planet was mathematically described by having it move along a circle with constant speed while the center of the circle also moved with constant speed on another circle. The center of this latter circle was either the Sun or did itself move on a circle around the Sun. However, merely by using the Sun where Hipparchus and Ptolemy had used the Earth, Copernicus found that he was able to reduce the total number of circles involved from seventy seven to thirty one. Later, to secure better accord with observations, he refined this idea somewhat by putting the sun near, but not quite at, the center of some of these aggregations of circles.

When Copernicus surveyed the extraordinary mathematical simplification which the heliocentric hypothesis afforded, his satisfaction and enthusiasm were unbounded. He had found a simpler mathematical account of the motions of the heavens and, hence, one which must therefore be preferred, for Copernicus, like all scientists of the Renaissance, was convinced that, "Nature is pleased with simplicity, and affects not the pomp of superfluous causes". God had so designed the universe. Copernicus could pride himself, too, that he had dared to think through what others, including Archimedes, had rejected as absurd.

Copernicus did not finish the job he set out to do. Though the hypothesis of a stationary sun considerably simplified astronomical theory and calculations, the epicyclic paths of the planets did not quite fit observations and Copernicus's few attempts to patch up his theory, always on the basis of circular motions, did not succeed.

It remained for the German, Johann Kepler, some fifty years later, to complete and extend the work of Copernicus. Like most youths of those days who showed some interest in learning he was headed for the ministry. While studying at the University of Tubingen he obtained private lessons in Copernican theory from a teacher with

whom he had become friendly. The simplicity of this theory impressed Kepler very much. Perhaps this leaning awakened suspicions in the superiors of the Lutheran Church, for they questioned Kepler's devoutness, cut short his ministerial career, and assigned him to the professorship of Mathematics and Morals at the University of Gratz. This position called for a knowledge of astrology, and so he set out to master the rules of that "art". By way of practice he checked its predictions with his own fortunes.

As an extracurricular activity he applied mathematics to matrimony. While at Gratz he had married a wealthy heiress. When this wife died he listed the young ladies eligible for the vacancy, rated each on a series of qualities, and averaged the grades. Women being notoriously less rational than nature, the highest ranking prospect refused to obey the dictates of mathematics and declined the honor of being Mrs. Kepler. Only by substituting a smaller numerical value was he able to balance the equation of matrimony.

Kepler's interest in astronomy continued and he left Gratz to become an assistant to that most famous observer, Tycho Brahe. On Brahe's death Kepler succeeded him as astronomer, part of his duties being once again of an astrological nature for he was required to cast horoscopes for worthies at the court of his employer, Rudolph II. He reconciled himself to this work with the philosophical view that nature provided astronomers with astrology just as she had provided all animals with a means of existence. He was wont to refer to astrology as the daughter of astronomy who nursed her own mother.

During the years he spent as astronomer to the Emperor Rudolph II, Kepler did his most serious work. It is extremely interesting that neither he nor Copernicus ever succeeded in ridding himself of the scholasticism from which their age was emerging, Kepler, in particular, mingled science and mathematics with theology and mysticism in his approach to astronomy, just as he combined wonderful imaginative power with meticulous care and extraordinary patience.

Moved by the beauty and harmonious relationships of the Copernican system he decided to devote himself to the search for whatever additional geometrical harmonies the data supplied by Tycho Brahe's observations might suggest, and, beyond that, to find the mathematical relationships binding all the phenomena of nature to each other.

However, his predilection for fitting the universe into a preconceived mathematical pattern led him to spend years in following up false trails. In the preface to his ***MYSTERY OF THE COSMOS*** we find him writing:

I undertake to prove that God, in creating the universe and regulating the order of the cosmos, had in view the five regular bodies of geometry as known since the days of Pythagoras and Plato, and that he has fixed according to those dimensions, the number of heavens, their proportions, and the relations of their movements.

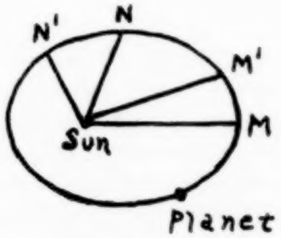
And so he postulated that the radii of the orbits of the planets were the radii of spheres related to the five regular solids in the following way. The largest radius was that of the orbit of Saturn. In a sphere of this radius he supposed a cube to be inscribed. In this cube a sphere was inscribed whose radius was that of the orbit of Jupiter. In this sphere he supposed a tetrahedron to be inscribed and in this another sphere whose radius was that of the orbit of Mars, and so on through the five regular solids. The scheme called for six spheres, just enough for the number of planets then known. The beauty and neatness of the scheme overwhelmed him so completely that he insisted for some time on the existence of just six planets because there were only five regular solids to determine the distances between them.

While publication of this "scientific" hypothesis brought fame to Kepler and makes fascinating reading even today, the deductions from it were, unfortunately, not in accord with observations. He reluctantly abandoned the idea, but not before he had made extraordinary efforts to apply it in modified form.

If the attempt to use the five regular solids to ferret out nature's secrets did not succeed, Kepler was eminently successful in later efforts to find harmonious mathematical relations. His most famous and important results are known today as Kepler's three laws of planetary motion. These laws became so famous and so valuable to science that he earned for himself the title of "legislator of the sky."

The first of these laws says that the path of each planet is not a circle but an ellipse with the sun at one focus of the ellipse. Hence one ellipse replaced the several circular motions superimposed on each other which even Copernicus's epicyclic theory had required to describe the motion of a planet. The simplicity gained thereby convinced Kepler that he must abandon attempts to use circular motions.

Kepler's second law is best understood with the aid of a diagram, *(Figure 1)*. Copernicus, we saw, believed that each planet moves on its immediate circle at a constant speed, that is, that it covers equal distances in equal intervals of time. Kepler at first held firmly to the doctrine that each planet moves along its ellipse at a constant speed, but his observations finally compelled him to abandon this cherished belief. His joy was great when



he discovered that he could replace it by an equally pleasing law, for thereby his conviction was reaffirmed that nature was mathematically designed.

If *MM'* and *NN'* *(Figure 1)* are distances traversed by a planet in equal intervals of *time*, then, according to the principle of constant speed, *MM'* and *NN'* would have to be equal distances. However, according to Kepler's second law, *MM'* and *NN'* are generally not equal but the *areas OMM'* and *ONN'* are equal. Thus Kepler replaced equal distances by equal areas and the mathematical rationality of the universe remained unshaken. To wrest such a secret from the heavens was indeed a triumph, for the relationship described is by no means as easily discernible as it may appear to be here on paper. These two laws Kepler published in a book entitled *ON THE MOTIONS OF THE PLANET MARS.*

Kepler's third law is as famous as his first two. It says that the square of the time of revolution of any planet is equal to the cube of its average distance from the sun (provided the time of the earth's revolution and the earth's distance from the sun are the units of time and distance.)

It is clear that mathematical concepts and mathematical laws are the essence of the new theory. But what is even more pertinent is that the mathematical excellence of the new theory recommended it to Copernicus and Kepler despite many most weighty arguments against it. Indeed had Copernicus or Kepler been less the mathematician and more the scientist, or had they been blind religionists, or even what the world calls "sensible" men, they could never have stood their ground. As scientist neither could answer Ptolemy's logical objections to a moving earth. Why, for example, if the earth rotates from west to east does not an object thrown up into the air fall back to the west of its original position? If, as all scientists since Greek times believed, the motion of an object is proportional to its mass, why doesn't the earth leave behind it objects of lesser weight? Why doesn't the earth's rotation cause objects on it to fly off into space just as an object whirled at the end of a string tends to fly off into space? These objections remained unanswered even at the time of Kepler's death.

No less a personage than Francis Bacon, the father of empirical science, summed up in 1622 the scientific arguments against Copernicanism:

In the system of Copernicus there are found many and great inconveniences: for both the loading of the earth with a triple motion is very incommodius and the separation of the sun from the company of the planets with which it has so many passions in common is likewise a difficulty and the introduction of so much immobility into nature by representing the sun and the stars are immovable........ all these are the speculations of one, who cares not what fictions he introduces into nature, provided his calculations answer.

While the clarity of Bacon's arguments could be surpassed, the opposition of a man of his reputation and ability could not be lightly brushed aside. Bacon's conservatism was due, incidentally, to his persistent inability to appreciate the importance of exact measurement in spite of his insistence on observation.

Were Copernicus and Kepler more "sensible", "practical", men they would never have defied their senses. We do not feel either the rotation or the revolution of the earth despite the fact that Copernican theory has us rotating through about 3/10 of a mile per second and revolving around the sun at the rate of about 18 miles per second. On the other hand we do see the motion of the sun. To the famous astronomical observer, Tycho Brahe, these arguments were conclusive proof that the earth must be stationary. In the words of Henry More, "sense pleads for Ptolemee."

Were Copernicus and Kepler religionists, they would not have been willing even to investigate the possibilities of a heliocentric hypothesis. Medieval theology, buttressed by the Ptolemaic system, held that man was at the center of the universe and the chief object of God's attention. By putting the sun at the center of the universe, the heliocentric theory denied this comforting dogma of the Church. It made man appear to be one of a possible host of wanderers on many planets drifting through a cold sky and around a burning ball. He was an insignificant speck of dust on a whirling globe instead of chief actor on the central stage. Unlikely was it that he was born to live gloriously and to attain paradise upon his death, or that he was the object of God's ministrations. Banished from the sky was the seat of God, the destination of saints and of a Deity ascended from the earth, and the paradise to which the good people could aspire. In short, the undermining of the Ptolemaic order of the universe removed a cornerstone of the Catholic edifice and threatened to topple the whole structure.

Copernicus's willingness to battle entrenched religious thinking is well evidenced by a passage in a letter to Pope Paul III:

If perhaps there are babblers [he says] who, although completely ignorant of mathematics, nevertheless take it upon themselves to pass judgment on mathematical questions and, improperly distorting some passages of the Scriptures to their purpose, dare to find fault with my system and censure it, I disregard them even to the extent of despising their judgment as uninformed.

Religion, physical science, and common sense bowed to mathematics at the behest of Copernicus and Kepler. So did even astronomy. The hypothesis of a moving earth calls for motion of the stars relative to the earth. But observations of the sixteenth century failed to detect this relative motion. Now no scientific hypothesis which is inconsistent with even one fact is really tenable. Nevertheless Copernicus and Kepler held to their heliocentric view. These moon-

struck lovers of mathematics were designing a beautiful theory. If the theory didn't fit all the facts, too bad for the facts. Copernicus, though deliberately vague on the question of the motion of the earth relative to the stars, disposed of the problem by stating that the stars were at an infinite distance. Apparently not too satisfied himself with this statement he assigned the problem to the philosophers. The true explanation, namely, that the stars were very far from the earth, so far as to render their relative motion undetectable, was not acceptable to the Renaissance "Greeks" who still believed in a closed and limited universe. The true distances involved were utterly beyond any figure which they would have thought reasonable. Actually the problem of accounting for the motion of the stars was not solved until 1838 when the mathematician Bessel finally measured the parallax of the nearest star and found it to be *0.76"*.

In view of all these arguments and forces working against the new theory why did Copernicus and Kepler advocate it? To answer this question we must examine their philosophical position.

It is necessary to dispose, first of all, of a commonly held belief that practical problems motivate and determine the course of mathematical and scientific thinking. Knowing that the great explorations of the fifteenth and sixteenth centuries demanded a better astronomy one is tempted to ascribe the motivation for their work to the need for more reliable geographical information and improved techniques in navigation. But Copernicus and Kepler were not at all concerned with the pressing practical problems of their age. What these men do owe to their times was the opportunity to come into contact with Greek thought, an opportunity furnished by the revival of learning in Italy. Copernicus, we saw, studied there and Kepler benefitted by Copernicus's work. Also both men owe to their times an atmosphere certainly more favorable to the acceptance of new ideas than the one which prevailed two centuries earlier. The geographical explorations, the Protestant Revolution, and so many other exciting movements were challenging conservatism and complacency, that one new theory did not have to bear the brunt of the natural opposition to change.

Actually Copernicus and Kepler investigated their most revolutionary theory to satisfy certain philosophical and religious interests. Having become convinced of the Pythagorean doctrine that the universe is a systematic, harmonious structure whose essence is mathematical law, they set about discovering this essence. Copernicus's published works give unmistakable, if indirect, indications of his reasons for devoting himself to astronomy. He values his theory of planetary motion not because it improves navigational procedures but because it reveals the true harmony, symmetry, and design in the divine workshop. It is wonderful and

overpowering evidence of God's presence. Writing of his achievement, which was thirty years in the making, Copernicus expresses his gratification:

> *We find, therefore, under this orderly arrangement, a wonderful symmetry in the universe, and a definite relation of harmony in the motion and magnitude of the orbs, of a kind that it is not possible to obtain in any other way.*

He does mention in the preface to the work already mentioned that he was asked by the Lateran Council to help in reforming the calendar which had become deranged over a period of many centuries. Though he writes that he kept this problem in mind it is quite apparent that it never dominated his thinking.

Kepler, too, makes clear his dearest interests. His published work, the fruit of his labors, attests to the sincerity of his search for harmony and law in the creations of the divine power. In the preface to his *MYSTERY OF THE COSMOS* he says:

> *Happy the man who devotes himself to the study of the heavens: he learns to set less value on what the world admires the most; the works of God are for him above all else, and their study will furnish him with the purest of enjoyments.*

A major treatise entitled *THE HARMONY OF THE WORLD*, which Kepler published in 1619, actually expounds a system of heavenly harmonies, a new "music of the spheres", which makes use of the varying velocities of the six planets. These harmonies are enjoyed by the sun which Kepler endowed with a soul specifically for this purpose. Lest it be supposed that this treatise was just a lapse into poetic mysticism, one should note that it also announced his celebrated third law of motion.

The work of Copernicus and Kepler is the work of men searching the universe for the harmony which their commingled religious and scientific beliefs assured them must exist, andexist in aesthetically satisfying mathematical form.

Having discovered newmathematical laws of the universe Copernicus and Kepler were irresistibly attracted by their simplicity. To these men, who were convinced that an omnipotent being designing a mathematical universe would necessarily prefer simplicity, this feature of the new theory attested to its truth. Indeed only a mathematician assured that the universe was rationally and simply ordered would have had the mental fortitude to set at naught the prevailing philosophical, religious, and scientific beliefs and the perseverance to work out the mathematics of so revolutionary an astronomy. Only one possessed of unshakeable convictions as to the importance of mathematics in the design of the universe would have dared to uphold the new theory against the powerful opposition it was sure to encounter. As a matter of history Copernicus did address himself to mathematicians because he expected that only these would

understand him, and in this respect he was not disappointed.

Granted that it was the superior mathematics of the new theory which inspired Copernicus and Kepler, and later Galileo, to repudiate religious convictions, scientific arguments, common sense, and well entrenched habits of thought, how did the theory help to shape modern times?

First of all, Copernican theory has done more to determine the content of modern science than is generally recognized. The most powerful and most useful single law of science is Newton's law of gravitation. Without undertaking here the discussion reserved for a more appropriate place we can say that the best experimental evidence for this law, the evidence which established it, depends entirely upon the heliocentric theory.

Secondly, Copernican theory is responsible for a new trend in science and human thought, barely perceived at the time but all important today. Since one's eyes do not see, nor one's body feel, the rotation and revolution of the earth, the new theory rejected the evidence of the senses and based itself on reason alone. Copernicus and Kepler thereby set the precedent which guides modern science, namely, that reason and mathematics are more important in understanding and interpreting the universe than the evidence of the senses. Vast portions of electrical and atomic theory and the whole theory of relativity would never have been conceived had scientists not come to accept the reliance upon reason which Copernican theory first exemplified. In this very significant sense Copernicus and Kepler begin the Age of Reason, while they also fulfilled one of the cardinal functions of scientists and mathematicians, namely, to provide a rational comprehension of natural phenomena.

By deflating the stock of *homo sapiens* Copernican theory reopened questions which the guardians of Western civilization were answering dogmatically upon the basis of Christian theology. Once there was only one answer; now there are ten or twenty to such basic questions as: *Why does man desire to live and for what purpose? Why should he be moral and principled? Why seek to preserve the race?* It is one thing for man to answer such questions in the belief that he is the child and ward of a generous, powerful, and provident God. It is another to answer them knowing that he is a speck of dust in a cyclone.

Copernican theory flung such questions in the faces of all thinking men and women, and thinking beings could not reject the challenge. Their struggles to recover mental equilibrium which was even further upset by the mathematical and scientific work following Copernicus and Kepler provide the key to the history of thought of the last few centuries.

Much evidence can be found in literature since Kepler's times of the agitation aroused by the new disturbing thoughts. The

metaphysical John Donne, though trained in and content with the encyclopedic and systematic scholasticism is compelled to acknowledge the undesirable complexity to which Ptolemaic theory had led:

> *We think the heavens enjoy their spherical*
> *Their round proportion, embracing all;*
> *But yet their various and perplexed course,*
> *Observed in divers ages, doth enforce*
> *Men to find out so many eccentric parts,*
> *Such diverse downright lines, such overthwarts,*
> *As disproportion that pure form.*

Though the argument for Copernicanism is clear to him he can only deplore that fact that the sun and stars no longer run in circles around the earth.

Milton, also, ponders the challenge to Ptolemaic theory but manages to retain it nevertheless. Unable to meet the new mathematics on its own ground he turns instead to rebuking its creators. Man should admire, not question, the works of God.

> *From Man or Angel the great Architect*
> *Did wisely to conceal, and not divulge*
> *His secrets to be scann'd by them who ought*
> *Rather admire;...........*

Yet even Milton was unconsciously moved to accept a more mysterious and a vaster space than the compact, thoroughly defined space of Dante, for example.

The gentle remonstrations of the milder poets, Ben Jonson's satire, Bacon's scientific arguments as well as personal jealousy, the ridicule of professors, the mathematical arguments of the brilliant Cardan, the resentment of astrologers who feared for their livelihood, Montaigne's scepticism, complete rejection from Shakespeare, and condescending mention from John Milton earned for Copernicus a reputation as a new Duns Scotus, the learned crazy one. In 1597 Galileo wrote to Kepler describing Copernicus as one "who though he has obtained immortal fame among the few, is nevertheless, ridiculed and hissed by the many, who are fools".

Nevertheless the opinion of the few prevailed. The cultural revolution gained momentum: people were compelled to think, to challenge existing dogmas, and to re-examine long accepted beliefs. And from criticism and re-examination of long established doctrines emerged many of the philosophical, religious, and ethical principles now accepted in Western civilization.

By far the greatest value of the heliocentric theory to modern times is the contribution it made to the battle for freedom of thought and expression. Because man is conservative, a creature

of habit, and convinced of his own importance, the new theory was decidedly unwelcome. The vested interests of well entrenched scholars and religious leaders caused them to oppose it. The most momentous battle in history, the battle for the freedom of the human mind, was joined on the issue of the right to advocate heliocentrism.

The self-appointed representatives of God entered the fray with vicious attacks on Copernicanism. Martin Luther called Copernicus an "upstart astrologer" and a "fool who wishes to reverse the entire science of astronomy". Calvin thundered: "Who will venture to place the authority of Copernicus above that of the Holy Spirit?" Do not Scriptures say that Joshua commanded the sun and not the earth to stand still? That the sun runs from one end of the heavens to the other? That the foundations of the earth are fixed and cannot be moved? The Inquisition condemned the theory as "that false Pythagorean doctrine utterly contrary to the Holy Scriptures," and in 1616 the Index banned all publications dealing with Copernicanism. Indeed if the fury and high office of the opposition is a good indication of the importance of an idea no more valuable one was ever advanced.

So shackled did the spirit of inquiry become in that age that when Galileo discovered the four satellites of Jupiter with his small telescope, some religionists refused to look through his instrument to see those bodies for themselves. And many who did tempt the devil by looking refused to believe their own eyes. It was this bigoted attitude that made it dangerous to advocate the new theory. One risked the fate of Giordano Bruno who was put to death by the Inquisition "as mercifully as possible and without the shedding of blood," the horrible formula for burning a prisoner at the stake.

Despite the earlier ecclesiastic prohibition of works on Copernicanism, Pope Urban VIII did give Galileo permission to publish a book on the subject, for the Pope believed that there was no danger that any one would ever prove the new theory necessarily true. Accordingly, in 1632 Galileo published his *DIALOGUE ON THE TWO CHIEF SYSTEMS OF THE WORLD*, in which he compared the Ptolemaic and Copernican doctrines. To please the Church and so pass the censors he incorporated a preface to the effect that the heliocentric idea was only a product of the imagination. Unfortunately Galileo wrote too well and the Pope began to fear that the argument for Copernicanism, like a bomb wrapped in silver foil, could still do a great deal of damage to the faith. The Church aroused itself once more to do battle against a heresy "more scandalous, more detestable, and more pernicious to Christianity than any contained in the books of Calvin, of Luther, and of all other heretics put together." Galileo was again called by the Roman Inquisition and compelled on the threat of torture to declare: "The falsity of the Copernican system cannot be doubted, especially by us Catholics....."

Burning faggots, the wheel, the rack, the gallows, and other ingenious refinements of torture were more conductive to orthodoxy than to scientific progress. When he heard of Galileo's persecution, Descartes, who was a nervous and timid individual, refrained from advocating the new theory and actually destroyed one of his own works on it.

However the heliocentric theory became a powerful weapon with which to fight the suppression of free thought. The truth (at least to the seventeenth and eighteenth centuries) of the new theory and its incomparable simplicity won more and more adherents as people gradually realized that the teachings of religious leaders could be fallible. It soon became impossible for these leaders to impose their authority over all Europe and the way was prepared for freer thought in all spheres.

The import of this battle and its favorable outcome should not be lost to us. No modern dictatorship has done more to suppress free thought than did the religious and secular powers of sixteenth and seventeenth century Europe. No modern dictatorship has perpetrated more fiendish torture, terror, and murder to impose its will on its subjects. Those who still enjoy and those who have lost the freedoms so recently acquired in Western civilization cannot fail to appreciate how much was at stake in the battle to advance the heliocentric theory and how much we owe to the men of gigantic intellect and stout heart who carried the fight.

Fortunately for us the very fires which consumed the martyrs to free inquiry dispelled the darkness of the Middle Ages. The fight to establish the heliocentric theory broke the stranglehold which ecclesiasticism held on the minds of men. The mathematical argument proved more compelling than the theological one and the battle for the freedom to think, speak and write was finally won. The scientific Declaration of Independence is a collection of mathematical theorems.

New York University

(continued from page 176)

Advanced Mathematics in Physics and Engineering by Arthur Bronwell; McGraw-Hill Book Company; New York; 1953; pp. xvi + 475; $6.00

Physicists and engineers inevitably find that more and better mathematics lead to unity and economy of thought. In recent years a number of books of the type under consideration have appeared. The selection and treatment of the mathematical material is of necessity determined by the field of applications and the previous training of the intended readers.

This book by Bronwell is intended for students who have had courses in mathematics through the calculus. The book contains discussions of infinite series, elementary functions of a complex variable, differential equations including solution in series and accounts of the Bessel and Legendre functions, Fourier-trigonometric, -Bessel, and -Legendre expansions, the Fourier integral, Laplace transforms, vector analysis, and analytic functions of a complex variable including the evaluation of integrals and conformal mapping. The mathematics merges into the applied field with the treatment of elastic and electric oscillations, Lagrange's equations, the wave equation, heat flow, fluid dynamics, electromagnetic theory, and dynamic stability. There are problems at the end of each chapter, and the answers to some of them are given. The book would be more useful if these problems were more numerous and more comprehensive. The reviewer is of the opinion that the applied part of this book will be valuable to the students for whom it is written. Generally speaking, the mathematical arguments tend to plausibility rather than rigor. The reviewer distrusts the eventual value of the amount of this kind of reasoning which the author uses.

Corman E. Miller

FERMAT COEFFICIENTS

P. A. Piza

Let n and $c \leq n$ be positive integers and let the number denoted by the symbol $(n:c)$ be defined by the relations

$$(n:1) = (n:n) = 1, \qquad (n:c<1) = (n:c>n) = 0.$$

And for $n>1$ and $c>1$ let the general term be

$$(n:c) = (n-1:c-1)\,\frac{2n-c}{c}. \tag{1}$$

If we construct a table of these $(n:c)$ numbers for the first few values of n, we shall find that most of them are integers but some are fractions as appears forthwith by rows n and columns c:

n \ c	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
(1)	1																
(2)	1	1															
(3)	1	2	1														
(4)	1	3	10/3	1													
(5)	1	4	7	5	1												
(6)	1	5	12	14	7	1											
(7)	1	6	55/3	30	126/5	28/3	1										
(8)	1	7	26	55	66	42	12	1									
(9)	1	8	35	91	143	132	66	15	1								
(10)	1	9	136/3	140	273	1001/3	1716/7	99	55/3	1							
(11)	1	10	57	204	476	728	715	429	143	22	1						
(12)	1	11	70	285	3876/5	1428	1768	1430	715	1001/5	26	1					
(13)	1	12	253/3	385	1197	2584	3876	3978	24310/9	1144	273	91/3	1				
(14)	1	13	100	506	1771	4389	7752	9690	8398	4862	1768	364	35	1			
(15)	1	14	117	650	2530	7084	14421	21318	22610	16796	8398	2652	476	40	1		
(16)	1	15	406/3	819	3510	32890/3	25300	43263	163438/3	49742	352716/11	41990/3	3876	612	136/3	1	
(17)	1	16	155	1015	23751/5	16380	296010/7	82225	120175	653752/5	104006	58786	22610	38760/7	3876/5	51	1

TABLE OF FERMAT COEFFICIENTS

It has been shown* that for any integer x

$$\frac{x^{2n+1} - x}{2n+1} = \sum_{c=1}^{n} (n:c) \sum_{a=1}^{x-1} (a^2+a)^c.$$

Hence for $x=2$ we have

$$\frac{4^n - 1}{2n+1} = \sum_{c=1}^{n} (n:c)\, 2^{c-1}.$$

**See the author's recent article 'Sur les Puissances des Nombres Triangulaires' in MATHESIS (1950), LIX, pp. 145-155.*

Because of the obvious relation of these $(n:c)$ numbers with Fermat's Theorem which states that if $p=2n+1$ is a prime and if x is prime to p then x^p-x is divisible by p, we propose that $(n:c)$ numbers be named Fermat Coefficients, FC for short.

Not only do Fermat Coefficients form a triangular table similar to Pascal's triangle, but they are also related to binomial coefficients by the formula

$$(n:c) = \binom{2n-c}{c-1} / c.$$

Therefore we have the theorem: The Fermat Coefficient $(n:c)$ is an integer if the c-th binomial coefficient of order $2n-c$ is divisible by c.

The defining relation (1) permits the generation of FC's recurrently by downward slanting diagonals. They can also be generated by rows with the formula

$$(n:c) = (n:c-1)\ \frac{2(n-c+1)(2n-2c+3)}{c(2n-c+1)},$$

and by columns with

$$(n:c) = (n-1:c)\ \frac{(2n-c)(2n-c-1)}{2(n-c)(2n-2c+1)}.$$

By virtue of their definition and their binomial relation it follows that FC's possess the following properties:

$$(n:4) = \frac{(n-2)(n-3)(2n-5)}{6} = \sum_{a=1}^{n-3} a^2.$$

$$(3r-1:2r-1) = (3r:2r+1).$$

$$(3r:2r) = 2\ (3r:2r+1).$$

$$3(n:n-1) = n(n+1)/2 = \binom{n+1}{2}.$$

$$5(n:n-2) = \binom{n+2}{4}. \qquad 7(n:n-3) = \binom{n+3}{6}.$$

In general for any integers n and r

$$(2r+1)\ (n:n-r) = \binom{n+r}{2r}.$$

Also

$$3(n:3) = \sum_{a=2}^{2n-3} (a:2). \qquad 5(n:5) = \sum_{a=1}^{n-3} (2a+1:4).$$

$$(n:3) = (2n-4:2n-5).$$

$$(n:5) = (2n-7:2n-9).$$

And again for any integers n and r

$$(n:2r+1) = (2n-3r-1:\ 2n-4r-1).$$

Hence we find that the sequence of FC's in any odd column $c=2r+1$ is equal to the sequence of alternate terms (in odd columns) in the downward slanting diagonal starting at $(r+1:1)$.

Still we find that the sequence of the FC's in any even column $c=2r$ has a similar relation with the sequence of the alternate (even columned) terms in the downward slanting diagonal starting at $(r:1)$, expressed by the following formula:

$$(n:2r) = \frac{n-2r+1}{r}\ (2n-3r+1:\ 2n-4r+2).$$

Also if in any row n we add the FC's of order n, and separately the odd columned terms and the even columned terms we get from the table:

n	sum of all FC's	sum of odd terms	sum of even terms
1	1	1	0
2	2	1	1
3	4	2	2
4	8+ 1/3	4+ 1/3	4
5	18	9	9
6	40	20	20
7	90+ 13/15	45+ 8/15	45+ 1/3
8	210	105	105
9	492	246	246
10	1165+ 10/21	582+ 17/21	582+ 2/3
11	2786	1393	1393
12	6710+ 2/5	3355+ 1/5	3355+ 1/5
13	16266+ 7/9	8133+ 4/9	8133+ 1/3
14	39650	19825	19825
15	97108	48554	48554
16	238824 + 1/11	119412 + 1/11	119412
17	589521+ 11/25	294760+ 19/25	294760+ 19/25

Note that for $n>1$ when all the FC's of order n are integers, the sums of odd and even terms appear to be equal integers. Also that whenever some FC is a fraction, the sums of odd and even terms are either equal or unequal fractions, or one of them is a fraction.

If c is a power of two, $c=2^m$, all FC's in column c starting at $(2^m:2^m)$ are integers. This follows from

$$n = 2^m+a, \quad a > 0, \quad 2n-c = 2^m - 2a.$$

$$(n:2^m) = \frac{(2^m-2a)(2^m-2a-1)\ldots.(2a+2)}{(2^m)!} \; .$$

Now $(2^m)! = 2^{2^m-1}k$, where k is a product of odd numbers. The numerator contains the product of 2^{m-1} even numbers and therefore it is also at least divisible by 2^{2^m-1}.

Fermat Coefficients have been found to possess many intrinsic and interesting properties that concern the primality and divisibility of numbers, some of which we shall endeavor to present.

Examination of the FC table indicates that whenever $2n+1=p$ is a prime number, all the FC's of order n are integers. We will prove this statement when $c < n$ is an odd prime. Then there exist integers a and b such that

$$p = 2n+1 = ac + b, \quad a > 0, \quad c > b > 0 \; ,$$

$$p \equiv b \pmod{c}.$$

Since

$$2n-c = p-1-c, \quad \text{we have}$$

$$(n:c) = \binom{2n-c}{c-1}/c = \frac{(p-c-1)(p-c-2)\ldots[p-c-(c-1)]}{c!} \; .$$

It must be shown that one of the factors of the numerator is divisible by the odd prime number c, and it is obvious that, since $b < c$, one of the factors is

$$p-c-b \equiv p-b \equiv 0 \pmod{c}.$$

When c is composite, the proof that $(n:c)$ is an integer if $2n+1$ is a prime is somewhat more complicated but not difficult, and it is left to the interested reader.

On the other hand suppose now that $2n+1$ is divisible by the odd prime c so that

$$2n+1 = ac.$$

Then

$$2n-c = ac - 1 - c = bc - 1,$$

and we have

$$(n:c) = \binom{2n-c}{c-1}/c = \frac{(bc-1)(bc-2)\ldots[bc-(c-1)]}{c!} \; .$$

It is easily seen that among the $c-1$ factors of the numerator there is none divisible by the prime c. This is evident because the extremes of the numerator factorial are $bc-1$ and $(b-1)c+1$, which are just short by unity one way or the other of being multiples of the odd prime c.

It follows then that if $2n+1$ is composite and contains the prime factor c, the Fermat Coefficient $(n:c)$ is an irreducible fraction whose denominator is c.

However, if c is composite and contains a common prime factor with $2n+1$, it does not follow necessarily that $(n:c)$ is an irreducible fraction, as shown by the instance $(13:6) = 2584$, which is an integer in spite of the fact that $2.13+1 = 27$ and 6 have the common factor 3.

Lehmer has proved that the converse of Fermat's Theorem is not true. That is, if 4^n-1 is divisible by $2n+1$, this number is not necessarily a prime number. He has found an infinitude of such composite numbers which he has named pseudoprimes, the smallest of which is $341 = 11.31$. Nonetheless the FC's $(170:11)$ and $(170:31)$ are irreducible fractions whose denominators are respectively 11 and 31.

In the same manner that all FC's in row n are integers if $2n+1$ is a prime, it can be shown that all FC's in the upward slanting diagonal starting at $(n-1:1)$ must also be integers.

Again in the downward slanting diagonal starting at $(n+1:1)$ the first $2n$ FC's will be integers if $2n+1$ is a prime. An irreducible fraction will not appear in such a diagonal until we reach the $(2n+1)$th column where $(3n+1:2n+1)$ will be an irreducible fraction whose denominator is the prime $2n+1$.

In conclusion we have the following theorems concerning FC's and prime and composite numbers:

1. All FC's of order n are integers if $2n+1$ is a prime.
2. If c is aprime, $c < n$, and if the FC $(n:c)$ is an irreducible fraction whose denominator is c, then $2n+1$ is composite and divisible by c.
3. In every case $(n:c)$ is an integer if $2n+1$ and c are relatively prime or if c is a power of 2.
4. A condition necessary and sufficient for $2n+1$ to be a prime is that every FC of order n be an integer.
5. All FC's $(n-c:c)$ in the upward slanting diagonal starting at $(n-1:1)$ are integers if $2n+1$ is a prime.
6. All the first $2n$ FC's $(n+c:c)$ in the downward slanting diagonal starting at $(n+1:1)$ are integers if $2n+1$ is a prime.

It seems that the network of fractions and integers in the FC table provides a kind of vivid picture of prime and divisible numbers. The above theorems, when further studied and developed,

and still other facts that will be found about Fermat Coefficients, may perhaps shed new light in the study of the greatest of all unsolved mathematical problems, namely, the distribution of prime numbers among the integers.

San Juan, Puerto Rico

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.

PROPOSALS

187. *Proposed by B. K. Gold's Calculus Class, Los Angeles City College.*

A student differentiated $f(x) = \arcsin (x^2 - a^2)/(x^2 + a^2)$ and $g(x) = \arcsin x/\sqrt{x^2 + a^2}$ and noted that their derivatives were equal. He reasoned that the anti-derivatives of $f'(x)$ and $g'(x)$ must differ only by an additive constant. Show that this is true.

188. *Proposed by C. F. White, Naval Research Laboratory, Washington, D. C..*

Trisect a given quadrant of a circle with a draftsman's $45° - 45° - 90°$ triangle and T - square.

189. *Proposed by T. L. Miksa, Aurora, Illinois.*

$$\text{Integrate} \int (R^2 - z^2) \sin^{-1} \sqrt{\frac{R^2 - b^2 - z^2}{R^2 - z^2}}\, dz$$

190. *Proposed by C. W. Trigg, Los Angeles City College.*

(1) Show that a square envelope with edge a can be folded into a stable hexahedron with congruent triangular faces. Overlapping

is permitted. (2) Find the surface area of the hexahedron. (3) Find the volume of the hexahedron. (4) Find the radius of the sphere which touches all the faces.

191. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Find the sum $$\sum_{s=0}^{n} (-1)^s \left[\frac{1}{s+1} + \frac{1}{s+2} + \cdots + \frac{1}{s+n}\right]\binom{n}{s}$$

192. *Proposed by V. Thebault, Tennie, Sarthe, France.*

If A', B', C' are the symmetries of the vertices of a triangle ABC with respect to a fixed point, the circumcircles of the three triangles $AB'C'$, $BC'A'$ and $CA'B'$ have a point in common which lies on the circumcircle of triangle ABC.

193. *Proposed by Francis J. Weiss, Washington, D. C..*

In comparing the relative food value of certain foodstuffs or the nutritional adequacy of different diets it is customary to express contents of individual nutrients in fractions or percentages of scientifically determined dietary allowances. There are about 40 different nutrients that are considered essential for maintenance of health (carbohydrates, fats, 10 amino acids, 12 vitamins, and the rest minerals and trace minerals) the lack of a single one may eventually be fatal. Consequently, no matter how plentiful the other nutrients are, if a single nutrient for instance lysine or vitamin C or iron is lacking, the diet is inadequate and death may result. Thus the individual nutrients or their fractions are clearly ***multiplicative*** which means that its maintenance value is zero, if only a single factor becomes zero.

On the other hand there is no doubt that the individual nutrients contained in a certain quantity of food or in a particular diet are also ***additive*** inasmuch as they contribute their share of calories and substances needed for the body. Consequently, if one wishes to compare the food value of certain foods (one quart of milk, one egg, etc.) or of a certain diet (lumberman in Oregon, child in California), one would have to express the combined nutrient ratios in a single expression that would have to be both additive and multiplicative, for instance:

Recommended dietary allowances per day $= A + B + C + D + \ldots\ldots\ldots\ldots$ (1)

Actual food intake or food value per gram $= a + b + c + d + \ldots\ldots\ldots\ldots$ (2)

Food index figure F $= \frac{a}{A} \circ . \frac{b}{B} \circ . \frac{c}{C} \circ . \frac{d}{D} \circ . \ldots\ldots\ldots\ldots$ (3)

Simultaneous multiplication and addition is indicated by the symbol 0. for lack of a better one.

Is it possible to carry out such combined operations mathematically? Can a Food Index function

$$f\left[\frac{a}{A}, \frac{b}{B}, \frac{c}{C}, \ldots\right]$$

be developed which will possess the desired multiplicative and additive properties?

SOLUTIONS

Errata

Acknowledgement of the solution of Problem 159 *by William Small, Rochester, New York* was omitted in the Sept. - Oct., 1953 issue.

THE BROKEN TREE

167. (March 1953) *Proposed by L. R. Galebaugh, Lebanon, Pennsylvania.*

A vertical tree 124 feet high is standing on a hillside whose angle of declivity is unknown. The tree breaks in such a manner that it does not completely separate at the break and the top reaches the ground 52 feet from the base of the stump, measured along the sloping surface of the hillside. The horizontal distance from the base of the stump to the fallen part is 33 feet. How high is the stump?

I. Solution by Clinton E. Jones, Tennessee A and I State University, Nashville, Tennessee. From the figure, let CH = 124 ft., $CB = y$, $AB = h$, $AC = p$, $DC = x$, $DB = z$, hence $h = (124 - y)$ft., $x = 33$ ft. and y the unknown height of the stump.

In triangle ABC,

$$(1)\quad p^2 = h^2 + y^2 - 2hy \cos a$$

and in triangle DBC,

$$(2)\quad \cos a = \frac{y}{\sqrt{x^2 + y^2}}\ .$$

Eliminating cos a between equations (1) and (2), and substituting the values for p, h, and x, we get,

$$(3) \qquad \frac{y^2}{\sqrt{1089 + y^2}} = \frac{6336 - 124y + y^2}{124 - y}$$

Simplifying we obtain,

$$(4) \qquad Ay^4 + By^3 + Cy^2 + Dy + E = 0 \quad \text{, where}$$

$$\begin{aligned} A &= 13{,}761 \\ B &= -1{,}841{,}400 \\ C &= 70{,}689{,}168 \\ D &= -1{,}711{,}176{,}192 \\ E &= 43{,}717{,}791{,}744 \end{aligned}$$

The positive real root of this biquadratic equation which satisfies equations (1), (2), (3), and (4) is $y = 44$. Hence, the height of the stump is 44 feet.

II. Solution by Leon Bankoff, Los Angeles, California. In the two right triangles of Figure (1) by juggling *3 - 4 - 5* and *5 - 12 - 13* Pythagorean triangles it is possible to verify that the stump is 44 feet high. Two other fallen tree problem combinations are shown in Figures (2) and (3). Figure (3) was made a non-primitive multi-Pythagorean system so as to be more appropriate for the great redwood trees of California.

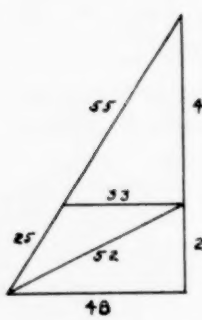


Fig 1

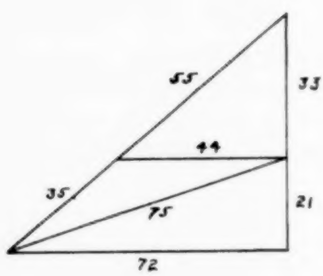


Fig 2

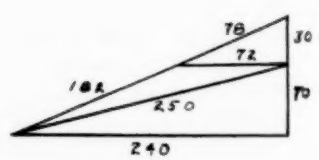


Fig 3

Also solved by *Leon Bankoff* (An analytic solution); *Sam Kravitz, East Cleveland, Ohio; and C. W. Trigg, Los Angeles City College (Two solutions).*

THE PUZZLED STUDENT

169. (May 1953) *Proposed by Norman Anning, University of Michigan.*

One student tries to adjust k to make $4x + y = 24$ a tangent to $y = kx^2$ and gets one answer. Another tries to make the same line a tangent to $x^2 = ky$ and gets two answers. Who is right and what is wrong?

Solution by M. Morduchow, Polytechnic Institute of Brooklyn.

The first student probably proceeded essentially as follows:

$$dy/dx = 2kx = -4. \quad \therefore x = -2/k, \quad y = k\ (-2/k)^2 = 4/k.$$

Substituting for x and y into $4x + y = 24$ leads to the (unique) result $k = -1/6$. This is correct, and can be verified qualitatively by a geometric sketch (although this type of verification is, strictly speaking, not essential here). The second student presumably proceeded essentially thus:

$$x = \pm\ (ky)^{1/2}, \quad dx/dy = \pm(1/2)k^{1/2}y^{-1/2} = -1/4.$$

Squaring both sides leads to $y = 4k$. $x = \pm 2k$.
Substituting into $4x + y = 24$ then leads to the two answers $k = 2$ (incorrect) or $k = -6$ (correct), according to whether the + sign or - sign, respectively, is used. The error here is due to the fact that at $x = 2k$, the derivative $dx/dy = k/2x = +1/4$, instead of $-1/4$. Hence, the point of tangency is $(-2k, 4k)$, and not $(\pm 2k, 4k)$. This leads to the single result $k = -6$.

This problem illustrates the care which must be taken in dealing with multiple-valued functions.

Several solvers noted that if $4x + y = 24$ and $x^2 = ky$ were combined to eliminate y then cleared of fractions, equating the discriminant to zero gave $k = -1/6$ and $k = 0$. However, this latter value of k is not admissable as it involves division by zero in the combined equation.

Also solved by *A. L. Epstein, Cambridge Research Center; Charles Salkind, Polytechnic Institute of Brooklyn; C. W. Trigg, Los Angeles City College, and the proposer.*

RADIUS OF ACTION

170. (May 1953) *Proposed by R. E. Horton, Lackland Air Force Base, Texas.*

A fighter plane flies under the following conditions:

Fuel Consumption: 100 gallons per hour

Fuel Capacity: Main tank holds 250 gallons
Two auxiliary wing tanks hold 100 gallons each.

Cruising air speed: Initial air speed is 400 miles per hour.
Air speed increases .125 mph per gallon of fuel consumed.
Air speed increases 5% when wing tanks are dropped simultaneously.

1) What is the effective range of the plane if a 20% reserve of fuel must be kept and the wing tanks are dropped simultaneously when both are empty?
2) What is the effective radius of action North with a wind of 30 mph from the South, assuming a fuel reserve of 20% and wing tanks dropped simultaneously when both are empty? (Radius of action is the distance a plane can fly and still return to its base).
3) Under what wind conditions will the time out exactly equal the time back on a radius of action problem for this plane?

Solution by B. K. Gold, Los Angeles City College.

1) Assuming the wing tanks are used up first they will be dropped after two hours. If a fuel reserve of 20% is kept, the total flying time is 3.6 hours. For the first two hours we have:

$$dv/dt = 12.5$$

so

$$v = 12.5t + 400$$

and

$$s = 6.25t^2 + 400t = 825 \text{ miles.}$$

Just after dropping the wing tanks

$$v = 446.25 \text{ mph.}$$

For the last 1.6 hours we have:

$$s = 6.25t^2 + 446.25t = 730 \text{ miles.}$$

So the effective range is 1555 miles.

2) If t_1 is the time flying north and $2 - t_1$ the amount of time flying south before dropping the wing tanks we have:

$$s\ (\text{out}) = 6.25t_1^2 + 400t_1 + 30t_1$$

$$s\ (\text{back}) = 6.25\ (2 - t_1)^2 + 400\ (2 - t_1) - 30\ (2 - t_1) +$$

$$+ 625 (1.6)^2 + 446.25 (1.6) - 30 (1.6)$$

As the distance out equals the distance back these are equal. Solving for t_1 the time flying north is approximately 1.75 hours which leads to a radius of action of approximately 771 miles.

3) Let w be the wind velocity from the south.
Then

$$s(\text{out}) = 6.25 (1.8)^2 + 1.8 (400 + w)$$

$$s(\text{back}) = 6.25 (.2)^2 + .2 (422.5 - w)$$

$$+ 6.25 (1.6)^2 + 1.6 (446.25 - w)$$

Equating and solving for w gives a wind velocity of approximately 21 miles per hour from the south.

Also solved by *Sam Kravitz, East Cleveland, Ohio (Partially), C. W. Trigg, Los Angeles City College, and the proposer.*

THE ORTHOGONAL CENTER

171. (May 1953) *Proposed by N. A. Court, University of Oklahoma.*

The four spheres having for great circles the polar circles of an orthocentric tetrahedron (T) have for orthogonal center the orthocenter of the medial tetrahedron of (T).

Solution by the proposer. The polar circles of the faces of an orthocentric tetrahedron (T) lie on the polar sphere (H) of (T). The four spheres having those circles for great circles have thus for orthogonal center the orthogonal conjugate of the center H of the sphere (H), (see the proposer's Modern Pure Solid Geometry, p 244, art 752), which point is the orthocenter of (T). Now the isogonal conjugate of H is the orthocenter of the medial tetrahedron of (T), (ibid, p 263, art 803), hence the proposition.

A LIMIT PROBLEM

172. (May 1953) *Proposed by H. E. Fettis, Wright-Patterson Air Force Base, Dayton, Ohio.*

Evaluate: $\lim_{a \to 0} (\pi^2 \csc^2 \pi a - 1/a^2)$.

I. Solution by Roy F. Reeves, Columbus, Ohio.

Let

$$L = \lim_{a\to 0} (\pi^2 \csc^2 \pi a - 1/a^2) = \lim_{a\to 0} \left[(a^2\pi^2\csc^2\pi a - 1)/a^2\right].$$

Apply l'Hospital's rule. Then,

$$L = \lim_{a\to 0} \left[(2a\pi^2\csc^2\pi a - \pi^3 a^2\csc^2\pi a\cdot\cot\pi a)/2a\right]$$

$$= \lim_{a\to 0} (2a\pi^2/2a) \cdot \lim_{a\to 0} \left[(\sin\pi a - \pi a\cos\pi a)/\sin^3\pi a\right]$$

$$= \lim_{a\to 0} \pi^2 \left[(\sin\pi a - \pi a\cos\pi a)/\sin^3\pi a\right]$$

Apply l'Hospital's rule again. Then,

$$L = \pi^2 \lim_{a\to 0} (\pi a/3\sin\pi a) = \pi^2/3.$$

II. Solution by J. Brandstatter, Los Angeles, California.

Write the limit as:

$$\lim_{a\to 0} \left[\frac{\pi^2 (\pi a) \csc^2\pi a - \pi^2}{(\pi a)^2}\right] \tag{1}$$

Let $x = \pi a$ and observe that the function $x \csc x$ is defined at the origin and has a valid power series expansion in a neighborhood including the origin. The function (1) may now be written as:

$$\frac{\pi^2(1 + x^2/6 + \dots)^2 - \pi^2}{x^2} = \frac{\pi^2(1 + x^2/3 + x^4/6^2 + \dots) - \pi^2}{x^2}$$

$$= \frac{\pi^2}{3} + \frac{\pi^2 x^2}{36} + \dots \text{, all}$$

operations on the series being valid because of absolute and uniform convergence of the series. By letting $a \to 0$ we obtain $\pi^2/3$ as the limit.

Also solved by *Leon Bankoff, Los Angeles, California; A. L. Epstein, Cambridge Research Center; Russell Freeman, Lexington, Kentucky; A. S. Gregory, Champaign, Illinois; J. R. Hatcher, Fisk University; John Jones Jr., Mississippi Southern College; M. S. Klamkin, Polytechnic Institute of Brooklyn; M. Morduchow, Polytechnic Institute of Brooklyn; L. A. Ringenberg, Eastern Illinois State College; Charles Salkind, Polytechnic Institute of Brooklyn; C. W. Trigg, Los Angeles City College; Artie Wells, University of Kentucky; and the proposer.*

Morduchow pointed out that the evaluation of an indeterminate form by expanding the pertinent functions in power series, retaining only the lowest powers necessary could also be used to evaluate such a limit as:

$$\lim_{a \to 0} \frac{(\pi a)^8 - \sin^8 \pi a}{a^4 \tan^6 \pi a} = 4/3\ \pi^4$$

Here one would have to evaluate the tenth derivatives of the numerator and denominator in order to use l'Hospital's Rule directly. He suggests greater emphasis on the series expansion method in elementary texts and classrooms.

DIOPHANTINE CUBICS

173. (May 1953) *Proposed by F. J. Duarte, Caracas, Venezuela.*

Prove that (1) the equation $x^3 - 6abx - 3ab(a + b) = 0$ has no solution in integers; (2) the equation $2x^3 - 6abx - 3ab(a + b) = 0$ has an infinite number of integer solutions.

I. Solution by the proposer. (1) In the equation

$$x^3 - 6abx - 3ab(a + b) = 0\ , \text{ let } a = \partial - \gamma\ ,\ b = \partial - \beta\ ,$$
$$x = \gamma + \beta - \partial\ .$$

Then we have:

$$(\gamma+\beta-\partial)^3 - 6(\partial-\gamma)(\partial-\beta)(\gamma+\beta-\partial) - 3(\partial-\gamma)(\partial-\beta)(2\partial-\gamma-\beta) = 0$$

or

$$\gamma^3 + \beta^3 - \partial^3 = 0$$

This equation has no solution in integers. Therefore the original equation has no solution in integers. (2) The equation

$$2x^3 - 6abx - 3ab(a + b) = 0$$

has a rational solution for $a = 2$, $b = 1$, and $x = 3$. Assume that $x = k$, $a = h$, and $b = 1$ where h and k are rational numbers. Let $x = k + t$ and $a = h + \rho t$. Then we have:

$$3t\left[2(k^2 - h) - \rho(2k + 2h + 1)\right] + t^2\left[2t - 3\rho(\rho + 2) + 6k\right] = 0$$

Setting each parenthetical expression equal to zero we have:

$$\rho = \frac{2(k^2 - h)}{2(k + h) + 1} \quad \text{and} \quad t = 3\left[\frac{\rho(\rho + 2)}{2} - k\right]$$

Now from the solution $x = k = 3$ and $a = h = 2$ we find that $\rho = 14/11$, $t = -333/121$, $x = 330/1331$ and $a = -2000/1331$. Thus we have the solutions:

$$\begin{array}{lll} x = 3 & a = 2 & b = 1 \\ x = 330 & a = -2000 & b = 1331 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

II. Solution by C. W. Trigg, Los Angeles City College.

(1) $x^3 - 6abx - 3ab(a + b) = 0$ may be written in the form $x^3 - 3ab(2x + a + b) = 0$. This suggests a simplifying substitution $a = m - n$, $b = m - p$, $x = n + p - m$, which reduces the equation to $n^3 + p^3 = m^3$. It is well-known that this equation has no solution in integers, so (1) has no solution.

(2) From (1) it is evident that $2x^3 - 6abx - 3ab(a + b) = 0$ may have a solution if $x^3 + n^3 + p^3 = m^3$ has. This last equation does have a two-parameter solution, namely

$$x = 1 - (u - 3v)(u^2 + 3v^2), \quad n = (u + 3v)(u^2 + 3v^2) - 1,$$

$$p = (u^2 + 3v^2)^2 - (u + 3v), \quad m = (u^2 + 3v^2)^2 - (u - 3v).$$

But

$$x = n + p - m, \quad \text{so } 3uv^2 - 3v + u^3 - 1 = 0, \quad \text{or}$$

$$v = \left[3 \pm \sqrt{9 + 12u - 12u^4}\right]/6u.$$

Thus $u = 1$ and $v = 0$ or 1. If $v = 0$ we have the trivial solution $a = b = x = 0$. If $v = 1$, then $x = 9$, $n = 15$, $p = 12$, $m = 18$, $a = 3$, $b = 6$. Thus there are an infinite number of solutions of (2), namely, $(a, b, x) = (k, 2k, 3k)$ or $(2k, k, 3k)$ where k is an integer.

Also solved by M. S. Klamkin, Polytechnic Institute of Brooklyn.

A SOLUTION IN PRIMES

174. (May 1953) *Proposed by J. E. Foster, Evanston, Illinois.*

The equation, $2^p + 1 = 3p'$, where p and p' are odd primes has been empirically confirmed for values of p through 17 as follows:

p	3	5	7	11	13	17
p'	3	11	43	683	2731	43691

Are there solutions of the equation for $p > 17$?

Solution by Sam Kravitz, East Cleveland, Ohio. From Lehmer's List of Primes it can be verified that when $p = 19$; $p' = 174{,}763$ and for $p = 23$, $p' = 2{,}796{,}203$ we have prime solutions for the equation.

Also solved by C. W. Trigg, Los Angeles City College who pointed out that Lucas stated (Assoc. franc. avanc. sc, 15, 1886, II, 191-2) that if n and $2n + 1$ are primes then $2n + 1$ is a factor of $2^n + 1$ when $n \equiv 1 \pmod 4$. In fact $2^{29} + 1 = 536{,}870{,}913 = (3)(59)(3{,}033{,}169)$ so p' is not prime for $p = 29$.

Editor's Note: *D. H. Lehmer, University of California, Berkeley,* has indicated that the following table gives all the known primes p' of the form $p' = (2^p + 1)/3$ where p is prime.

p	p'
3	3
5	11
7	43
11	683
13	2731
17	43691
19	1 74763
23	27 96203
31	7158 27883
43	293 20310 07403
61	768 61433 64045 64651
79	2014 87636 60243 81957 84363

The large prime $(2^{79} + 1)/3$ is due to A. Ferrier, MTAC v.4, 1950 p. 54.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 103. If $\cos 17A = f(\cos A)$, then $\sin 17A = f(\sin A)$. [*Submitted by Norman Anning.*]

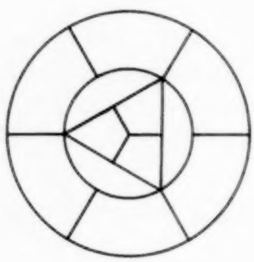
Q 104. Prove without searching for factors that the numbers *1,000,000,009* and *1,000,000,011* can not be paired primes of the form P and $P + 2$. [*Submitted by Arthur Parges.*]

Q 105. Show that

$$\sum_{p=1}^{n} [n^2 - (2p - 1)n] = 0.$$

[*Submitted by Fred Marer.*]

Q 106. Show that there is no path along the lines in the figure which passes through each junction point once and only once. [*Submitted by Leo Moser.*]



ANSWERS

A 103. In general if $\cos (4k + 1)A = f(\cos A)$ let $A = \pi/2 - B$. Then $\cos [(4k + 1)(\pi/2 - B)] = f [\cos (\pi/2 - B)] = f(\sin B)$. But $\cos [(4k + 1)(\pi/2 - B)] = \cos [\pi/2 - (4k + 1)B] = \sin (4k + 1)B$. Hence $\sin (4k + 1)A = f(\sin A)$.

A 104. Prime numbers are either of the form $6n + 1$ or $6n + 5$. The lesser of two paired primes cannot be of the first type since the greater would be of the form $6n + 3$, a composite number. Thus all paired primes are of the form $6n + 5$, $6n + 7$ and their sum is divisible by 12. Since the sum of the given numbers is not divisible by 3 they cannot be paired primes.

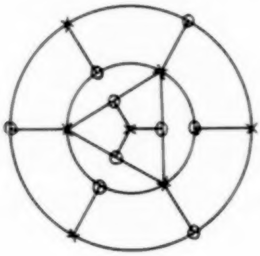
A 105.

$$\sum_{p=1}^{n} [n^2 - (2p - 1)n] = n \cdot n^2 - n \sum_{p=1}^{n} (2p - 1)$$

$$= n^3 - n [1 + 3 + 5 + \dots + (2n - 1)]$$

$$= n^3 - n \cdot n^2 = 0 .$$

A 106. We may divide the points into two classes, 7 "crossed" ones and 9 circled ones. Since any path which can be chosen leads from a point of one type to a point of the other type, it will be impossible to reach more than 8 of the circled points.



TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 10. Can a checker be moved from position (1, 1) to position (8, 8), moving one square at a time and never diagonally, in such a way that the checker enters each square once and only once? [*Submitted by M. S. Klamkin.*]

T 11. Three men rented a hotel room together, being charged \$30. The roomclerk later discovered that the correct room rate was \$25, so he sent a bellboy to the room with a \$5 refund. The larcenous bellboy decided to return only \$3 and to keep \$2 himself.

Now, each man paid \$10 originally and received a refund of \$1, so the cost per man was \$9. Three times \$9 is \$27, and the bellboy kept \$2, making a total of \$29. What became of the other dollar? [*Submitted by George Mapes.*]

T 12. At a certain subway station, uptown and downtown trains each arrive at 15-minute intervals on parallel tracks. The passengers, who arrive uniformly with respect to time (even arbitrarily small periods of time), board the first train coming into the station. How do you account for the fact that 4 times as many people travel uptown from this station as those that travel downtown? [*Submitted by M. S. Klamkin.*]

T 13. Let A, B, C be points on the x, y, z axes, respectively, (distinct from the origin). When will triangle ABC be a right triangle? [*Submitted by E. A. Nordhaus.*]

SOLUTIONS

S 10. Yes. The trick here is to move to (1, 2) or (2, 1) and then back to (1, 1) which is then *entered* for the first time. One possible sequence of moves is (1, 1), (1, 2), (1, 1), (2, 1), (2, 3), (1, 3), (1, 4), (2, 4), (2, 5), (1, 5), (1, 6), (2, 6), (2, 7), (1, 7), (1, 8), (3, 8), (3, 1), (4, 1), (4, 8), (5, 8), (5, 1), (6, 1), (6, 8), (7, 8), (7, 1), (8, 1), (8, 8).

S 11. There is no other dollar. If the men's expenditure be considered *positive*, then what the bellboy kept is *negative*. Otherwise, the men spent \$27, whereas the hotel kept \$25 and the bellboy \$2, a total of \$27.

S 12. The downtown trains arrive three minutes after the uptown trains, on a schedule something like 5:00 uptown, 5:03 downtown, 5:15 uptown, 5:18 downtown, ... Thus the probability of catching an uptown train is 12/15 or 4/5, and of catching a downtown train is 1/5.

S 13. Never. If

$$\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2,$$

then

$$\overline{OA}^2 + \overline{OB}^2 + \overline{OB}^2 + \overline{OC}^2 = \overline{OA}^2 + \overline{OC}^2,$$

or $\overline{OB} = 0$, a contradiction.

PYTHAGORAS, HIS THEOREM AND SOME GADGETS

James Clifton Eaves

The purpose of this note is to present two or three gadgets which may be used to demonstrate the Pythagorean Theorem. The models may easily be made of heavy paper, cardboard, or light weight plastics. Of course, more elaborate ones, large and small sized, can be constructed of wood, metal, wallboard, heavy plastics, etc. The first of these demonstrators is based upon the geometric figure suggested by Chou-pei, China, about 1100 B.C. Chou-pei gave no proof. He merely drew the figure and uttered not one murmur of explanation. The second model is based upon the well known figure usually accredited to Bhaskara. It has a novel attachment which renders the demonstration dependent only upon pulling a string, or turning a crank, or winding a key, whichever the maker has available. The proof for the third demonstrator is new, but has a slight similarity to proofs 19 and 23, pp. 112-114 of reference 9. The fourth treats the old familiar special case of the *3-4-5* triangle. It is interesting to see how the two small squares can be folded together to form the square on the hypotenuse.

These gadgets were selected because the proofs can be given by using the gadget figures themselves.

In presenting this topic there are two questions which seem to bear directly on the subject.

The first question: Demonstrative Geometry. There are those among us who advocate a demonstration of any kind (including tap dances in *2/4, 3/4, 4/4,* or *6/8* time) even at the expense of systematic proofs. Others in the mathematics camp frown and flinch at the slightest mention of any sort of demonstration. The writer, evading an answer to this issue, believes that a great mistake is made if a person devours nothing but demonstrations during his opportunity to learn geometry. On the other hand, if a model makes the problem clearer and more understandable, then the relatively small amount of time devoted to the demonstration is, perhaps, well spent. If one of these gadgets helps anyone to understand the meaning of the Pythagorean Theorem, if one helps him to get the real picture of the problem, and, if it helps him to outline and render a logical proof of this important and interesting proposition then the writer shall have no qualms about the time spent in designing these gadgets in answer to an inquiry for just one such tool.

The second question: Why the name "Pythagorean Theorem"? The Indian and Egyptian surveyors used the *3-4-5* triangle in their rope-stretching procedure long before Pythagoras, possibly as early as 4000 B.C. There is no evidence that these surveyors knew of the existence of any other Pythagorean triangle, the particular *3-4-5* being sufficient for their needs. The Chinese, Japanese and Hindus knew the theorem in a more general form perhaps a thousand years before the Christian Era. For years it was argued that the Babylonians knew the theorem in its entirety as early as 2000 B.C. It was, however, pointed out rather forcefully that, behind the argument, there was nothing but wishful thinking on the part of the pro-Babylonians or anti-Pythagoreans. The recently read Plimpton Library tablets, which lay for a number of years undeciphered and classified as commercial accounts, proved conclusively that the Babylonians knew many integral solutions of the equation $a^2 + b^2 = c^2$ and the order of record seems to bear out the contentions of many that the general solution $a = u^2-v^2$, $b = 2uv$, $c = u^2+v^2$, for the well known restrictions on u, v, was listed among their recorded facts.

Some seem to think that Pythagoras got the idea of the theorem from the simple *3-4-5* rope-stretchers' surveying triangle while he traveled and studied mysticism in Egypt. After founding his school at Crotona he studied and lectured on number theory as well as geometry. He actually gave a class of primitive solutions, namely, $a = 2n+1$, $b = 2n^2+2n$, $c = 2n^2+2n+1$, where a, b, c are the sides of the triangle. Often the solution accredited to Pythagoras is given the pattern

$$\frac{n^2-1}{2},\ n,\ \frac{n^2+1}{2},$$

for odd integer $n>1$.

There is a trace of the consideration of the problem, "When does the Pythagorean triangle have an integral altitude upon the hypotenuse?" The Pythagoreans investigated the problem, "Which Pythagorean triangles are isosceles?" Finding such triangles nonexistent, they invented irrationals only to drown Hippasus for divulging this Pythagorean secret.

Be it true or not true that Pythagoras should be given credit for having discovered this famous theorem it is probably safe to say that it is most likely to be known as the Pythagorean Theorem for at least another three or four years! He did more to develop it and bring it to light than anyone up to his time. Even if he had nothing to do with the problem it seems only natural and according to generally accepted procedures to refer to it in this manner.

When one recalls that it was de Moivre who established Stirling's factorial approximation, Stirling who discovered the special case

of Taylor's theorem now known as Maclaurin's theorem, Napier who generalized and revamped his first discovery to give us Briggian logs, Ruffini who gave us Horner's method sixteen years before Horner but six centuries after Ch'in Chiu-Shao, Waring who stated Cauchy's ratio test which was proved by Gauss who anticipated Bolyai by some thirty years and Lobatschewsky by about ten years in his discovery of Non-Euclidean Geometry, commonly called the Geometry of Lobatschewsky, then just what does one more slight discrepancy matter in the crediting to some mathematician a theorem which the man publicized so very much. The namesake is rightly deserved.

It is hoped that the following gismos may be interesting ways to remember correctly not only the Pythagorean Theorem but also a proof. If Bhaskara need say only "Behold" for the proof which he contributed, then, with this approach, Chou-Pei's silence was sufficient.

Gadget 1. The parts needed are as displayed below.

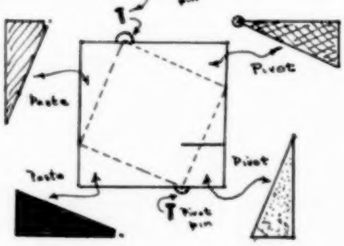


fig. 1.0

Assembled, the demonstration is given in four steps.

fig. 1.1

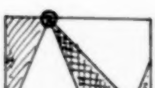


fig. 1.2

fig. 1.3

fig. 1.4

If one considers the "exposed", that is, unshaded area which forms the square on the hypotenuse of the black basic right triangle (fig. 1.1), then it is easy to see that the exposed area in figure 1.2 is equal to that of figure 1.1, that of figure 1.3 equal to that of figure 1.2, and finally, that of figure 1.4 equal to that of figure 1.3. The exposed area in figure 1.4 forms the two squares which *CAN* be constructed on the two legs of the basic triangle. This conforms to the often used statement of the theorem, viz.:

For any right triangle the square which can be constructed on the hypotenuse is equal to the sum of the squares which can be constructed on the two legs of the triangle.

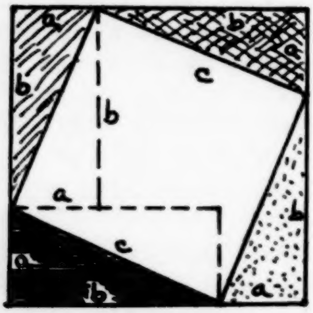


fig. 1.5

For a geometric proof we consider figure 1.5. All the triangles are congruent and, by the axiom of addition, the theorem follows.

For an algebraic proof, which is given today in some texts, we merely note, by considering the areas of the parts of the whole square, that

$$(a+b)^2 = 4\left(\frac{ab}{2}\right) + c^2 .$$

This equation reduces to $a^2 + b^2 = c^2$. When the algebraic proof is sought the theorem should be stated in a different manner from that given above.

Gadget 2. The parts needed, in relative position, are displayed below.

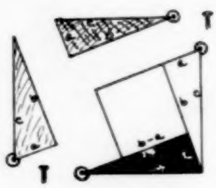


fig. 2.0

Assembled, the demonstration is given as follows:

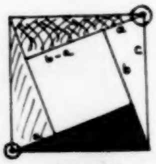


fig. 2.1

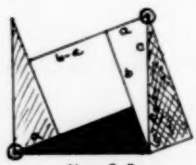


fig. 2.2

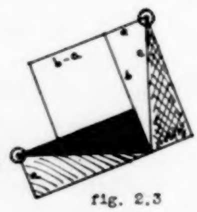


fig. 2.3

It is obvious that the large square, of side c, on the hypotenuse of the basic (black) triangle has been "unfolded" into two squares, one of side a and the other of side b.

fig. 2.4

For the well known algebraic proof we have, from fig. 2.4,

$$c^2 = (b-a)^2 + 4(\frac{ab}{2})$$

Which reduces to

$$c^2 = b^2 + a^2.$$

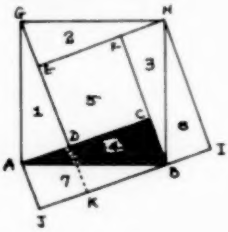


fig. 2.5

For a geometric proof we have, from fig. 2.5, $\triangle 6$ constructed with $HI \parallel FB$, $BI \parallel FH$. Similarly, we construct $\triangle 7$. Extend GD to K. It is easy to show that KH and AK are squares. Thus,

$$\square AH = \triangle 1 + \triangle 2 + \triangle 3 + \triangle 4 + \square 5 = \triangle 7 + \triangle 6 + \triangle 3 + \triangle 4 + \square 5 = \square AK + \square KH.$$

A novel twist can be added to Gadget 2. It appeals very much to those whose ability seems to lie in "turning a crank". If, as in fig. 2.6, the gadget is float-mounted on a heavy board, triangles A and B mounted to two large pulley discs connected with a third crank-turned center disc by cord or wire, then, by turning the crank behind the board, the triangles A and B can be made to revolve to the desired positions shown in fig. 2.3. They are returned to rest positions shown in fig. 2.6 by twist springs located at the pivot-vertices of these triangles.

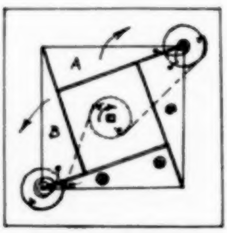


fig. 2.6

Gadget 3. We first present here the geometric figure together with the necessary constructions and outline of proof. Then, the disection and necessary coupling is indicated. Using the basic figure we do the following:

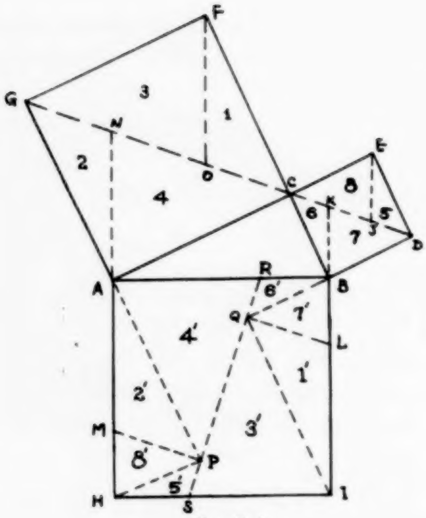


fig. 3.0

1. Draw GCD.
2. Extend AH, BI to N,K respectively.
3. Draw FO, EJ $||$ AH.
4. Extend DB to Q such that $QB = BD$.
5. Draw CQ and extend to S.
6. Draw QL $||$ CD.
7. Extend GA to P and draw PM $||$ GC.
8. Draw QI and PH.

It is easy to establish the following, in this order: $\triangle 6 \cong \triangle 6'$, $\triangle 7 \cong \triangle 7'$, $\triangle 2 \cong \triangle 2'$, $\triangle 4 \cong \triangle 4'$, $\triangle 1 \cong \triangle 1'$, $\triangle 3 \cong \triangle 3'$, $\triangle 8 \cong \triangle 8'$, $\triangle 5 \cong \triangle 5'$. This proves the theorem.

And now, the parts are cut and assembled in the following manner.

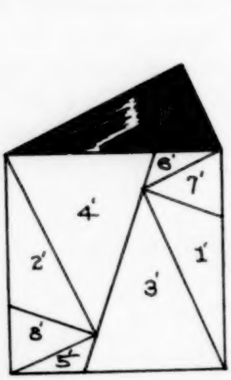


fig. 3.1

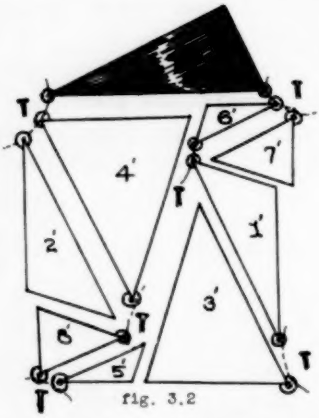


fig. 3.2

It is easy to unfold the hypotenuse square of fig. 3.2 to form the two squares on the legs of the triangle, fig. 3.0.

Gadget 4. The following illustrates how the 3×3, the 4×4 and then both, in the 3-4-5 case, may be cut. There are many disections possible, some of which require the "inter-weaving" of the 3×3 with the 4×4.

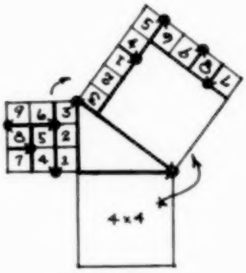


fig. 4.1

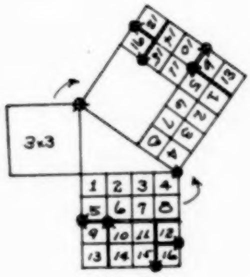


fig. 4.2

A few references which may be of interest in connection with these remarks are given.

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Alabama Polytechnic Institute

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

The following note is based on information furnished by Norman Anning, Professor Emeritus, University of Michigan, in response to an inquiry published inthis Vol., No. 1, of Mathematics Magazine. Ed.

CONCERNING THE RADIAN, SEE:

Footnote 7, *Enciclopedia Della Matematiche Elementari*, Vol. 2, part 1, page 546 (footnote)

"7) La parola *radiante* fu usata per la prima volta da J. Thomson, v. F. Cajori, *History of Mathematics*, 2^a ed., New York 1919, p. 484. G. B. Halsted (1853-1922) (*Mensuration*, Boston 1881, p. 83) propose l'uso della lettera ϱ per indicare il radiante, ma la proposta, come altre, non ha avuto fortuna."

A History of Mathematics by Cajori, 2nd Ed., Rev. and Enlarged, The Macmillan Company, 1929. (page 484).

"An isolated matter of interest is the origin of the term 'radian,' used with trigonometric functions. It first appeared in print on June 5, 1873, in examination questions set by James Thomson at Queen's College, Belfast. James Thomson was a brother of Lord Kelvin. He used the term as early as 1871, while in 1869 Thomas Muir, then of St. Andrew's University, hesitated between 'rad,' 'radial' and 'radian.' In 1874 T. Muir adopted 'radian' after a consultation with James Thomson.[1]

"[1] *Nature*, Vol. 83, pp. 156, 217, 459, 460."

Overheard: "$5^2 = 25$. We know that. But if you're an engineer and the slide rule reads $5^2 = 24.95$, then $5^2 = 24.95$, period."

GRAPHICAL GROUP REPRESENTATIONS

John E. Maxfield

INTRODUCTION The usual approach now used for the beginning student of Group theory is that of defining a group by a set of postulates. One then investigates some of the properties of groups and some of their representations. The Cayley multiplication table and the representations of groups as sets of permutations are usually all the beginning student becomes acquainted with. The more serious student continues with the matric representations of groups and special properties of certain classes of groups.

Some less well-known representations of groups are those of the Cayley group diagram and the color group. These form an interesting exercise for the student and give some insight into the structure of a given group.

SECTION I. In this section we will define our terms and build the foundations necessary to justify the group diagram and the color group.

Definition 1. A *group* is a mathematical system composed of elements, an equals relation, and one operation (x) satisfying the following restrictions.

1) The system is closed under the operation x, which is well defined (i.e. equals may be substituted for equals).

2) The elements are associative under the operation x. That is, $(a \; x \; b) \; x \; c = a \; x \; (b \; x \; c)$ for all a, b, c of the set.

3) There exists an identity element i such that $a \; x \; i = i \; x \; a = a$ for every element a of the set.

4) Every element a has an inverse a^{-1} such that $a^{-1} \; x \; a = a \; x \; a^{-1} = i$.

Definition 2. A *finite group* is a group consisting of only a finite number of elements. The number of elements in a finite group is called the order of the group.

Definition 3. The *Cayley multiplication table* for a group of order n is formed in the following way. One writes the elements of the group along the top and side of an n by n grid. Since multiplication is well defined relative to the equals relation one can fill the grid in with all the products indicated, the element of the side times the element on top. This table defines the group. An example follows for the Cayley multiplication table for the Quaternion group. In the table each different element is labeled with a lower case letter.

	I	S	S^2	S^3	T	T^3	TS	TS^3
I	I_a	S_b	S^2_c	S^3_d	T_e	T^3_f	TS_g	TS^3_h
S	S_b	S^2_c	S^3_d	I_a	TS^3_h	TS_g	T_e	T^3_f
S^2	S^2_c	S^3_d	I_a	S_b	T^3_f	T_e	TS^3_h	TS_g
S^3	S^3_d	I_a	S_b	S^2_c	TS_g	TS^3_h	T^3_f	T_e
T	T_e	TS_g	T^3_f	TS^3_h	S^2_c	I_a	S^3_d	S_b
T^3	T^3_f	TS^3_h	T_e	TS_g	I_a	S^2_c	S_b	S^3_d
TS	TS_g	T^3_f	TS^3_h	T_e	S_b	S^3_d	S^2_c	I_a
TS^3	TS^3_h	T_e	TS_g	T^3_f	S^3_d	S_b	I_a	S^2_c

figure 1

Definition 4. Let there be given n letters $a_1 \ldots a_n$. There are n factorial possible arrangements of these letters. The operation of rearranging this set of letters to another possible arrangement is called a *permutation.*

Example 1. $a_1a_2a_3$ and $a_2a_1a_3$ are two different arrangements of these three letters. The operation of rearranging the first to the second is indicated by the symbol $\begin{Bmatrix}123\\213\end{Bmatrix}$ which means that a_1 is replaced by a_2, a_2 by a_1, and a_3 by a_3.

Definition 5. A permutation which replaces

$$a_{i_1} \text{ by } a_{i_2},\ a_{i_2} \text{ by } a_{i_3},\ \ldots\ ,\ \text{and } a_{i_m} \text{ by } a_{i_1},$$

leaving all other letters unchanged is called a *cycle* of order m, written

$$(i_1, i_2, \ \ldots \ , i_m).$$

Theorem 1. Every permutation can be written as a product of cycles, no two of which have a letter in common.

Definition 6. A group of permutations of order n on n letters is called a *regular substitution group.*

Theorem 2. (Cayley) Every finite group can be represented as a regular substitution group.

Theorem 3. Every permutation of a regular substitution group is either a cycle or is composed of a number of cycles each of the same order.

Example 2. The following permutations written as products of cycles represent the Quaternion group. (1), $(abcd)(ehfg)$, $(ac)(bd)(ef)(gh)$, $(adcb)(egfh)$, $(aecf)(bgdh)$, $(afce)(bhdg)$, $(agch)(bfde)$, and $(ahcg)(bedf)$.

In order to find say $(abcd)(ehfg)$, the second element, form the permutation $\binom{abcdefgh}{bcdahgef}$ the upper line being the lower case letters in figure 1 in their natural order, the second line representing the lower case letters of the line obtained by multiplying on the left by the second element (S). Then $\binom{abcdefgh}{bcdahgef} = (abcd)(ehfg)$.

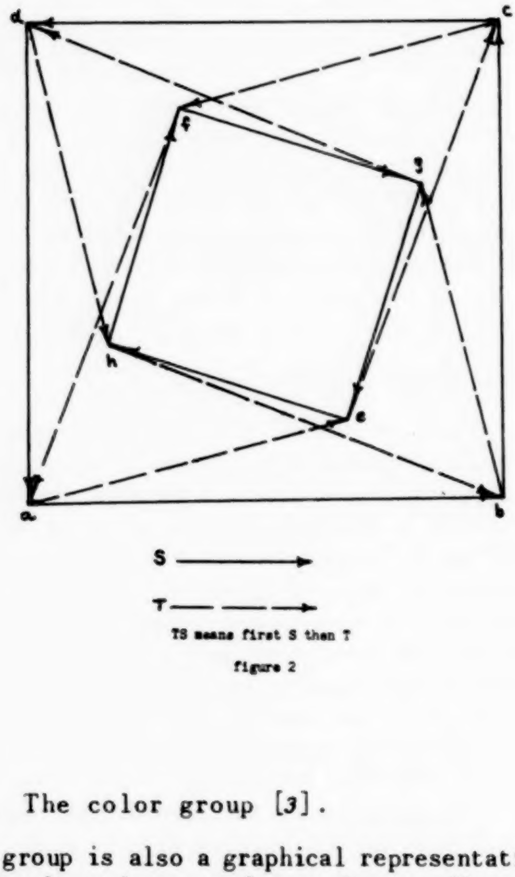
Definition 7. A ***primitive set of permutations*** of a group is a set of permutations such that every permutation of the group can be represented as a product of these permutations but that no one of them can be represented in terms of the others by taking products.

Example 3. A set of primitive permutations of the Quaternion group are the following: $(abcd)(ehfg) = S$, and $(aecf)(bgdh) = T$. The elements in Ex. 2 are then in order, I, S, S^2, S^3, T, T^3, TS, and TS^3.

SECTION II. The group diagram [*1*] , [*2*] .

Consider any primitive set of permutations $p_1p_2 \ldots p_s$ of a group G of order n. Consider a set of n points $a \ldots n$. A change of a into b will be represented by a directed line joining the two points. Thus a cycle of order t will be a t-gon. A cycle of order 2 will be a 2-gon or two points joined by a double headed arrow. From each point one line representing each primitive substitution both enters and leaves. Note that if there is a 2-cycle among the primitive cycles there will be from each point only one line representing that 2-cycle, since such a line both enters and leaves a point. The fact that every finite group has such a representation follows from Theorems 2 and 3. The proof that such a diagram represents a group is quite simple. It consists in noting that if one considers the set $(a, b, \ldots, n) = \alpha$ and applies the operations representing the primitive permutations one transforms α into $p_i\alpha$. Thus applying consecutively the operations on α one constructs the permutations of α representing the group G.

Example 4. The group diagram for the Quaternion group follows.



TS means first S then T

figure 2

SECTION III. The color group [3].

The color group is also a graphical representation of a group. One can consider it to be a supplemented group diagram. In order to form a color group start with the group diagram of a given group and add further directed lines representing elements of the group that are not already represented and whose inverses are not already represented. Continue this process until every element in the group is represented by a directed line or by going in the opposite sense along a directed line (i.e. its inverse is represented by a directed line). Thus, an element and its inverse are represented by the same "color". There are, then, $n(n-1)/2$ different lines on the figure.

Example 5. The color group for the Quaternion group follows.

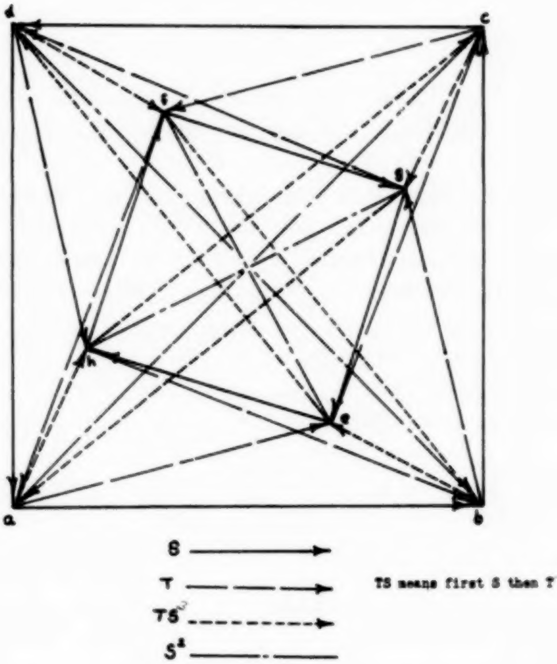


figure 3

The main interest that is generated by the color group over the group diagram is that the subgroups of the group can easily be selected from the color group by picking closed figures that satisfy the conditions that every point has all the elements in the figure both entering and leaving and that can be started at any point of the diagram.

Example 6. The proper subgroups (not Q nor I) of the Quaternion group follow.

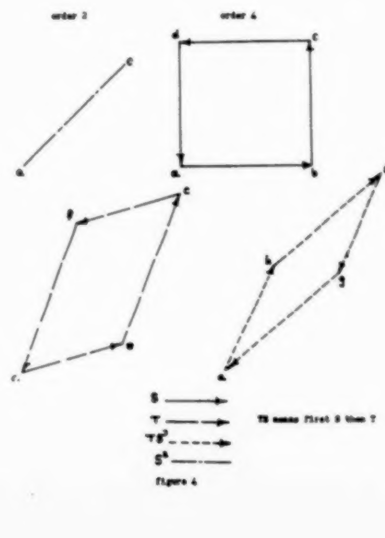


figure 4

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REVIEW OF LEWIS AND LANGFORD: *SYMBOLIC LOGIC*

Not for the beginner but a treasury of insights into modern logical theory for the mature student, this reprint without alteration of a book which has been for two decades among the most frequently cited works on symbolic logic is a welcome event. Since Lewis's earlier and now long out-of-print *A Survey of Symbolic Logic*, the present book is one of the few readily available which treat modern logic in historical perspective; a chapter is devoted, in this connection, to the early work of Leibniz, Hamilton, De Morgan, Boole, Jevons, Peirce, and Peano. Four more chapters provide a rarely-to-be-found account of the Boolean and Boole-Schroder algebras and their interpretations--topics of revived interest in the light of their recent exploitation by Claude Shannon in the analysis of relay and switching circuits for the Bell Telephone Company. In the eight chapters remaining, the authors provide logistic foundation for the calculi of propositions and of functions and explore truth-value systems, the logic of modalities, postulational technique, and the antinomies. Several of these subjects are discussed from the highly original point of view of Lewis's own *system of strict implication.*

Implication, surely the heart of deductive logic, has long puzzled logicians. The puzzles arise from the common practice of using the term 'implication' to denote several distinct relations. Consider: (1) If sugar is bitter then the moon is made of green cheese. (2) If there is smoke then there is fire. (3) If there was wind and rain today then there was wind today. Exhibited here are three quite different if-then statements plausibly termed

'implications'. In the latter two there is a sweet reasonableness lacking altogether in the first. The physicist, presumably, is interested primarily in the second sort of implication, the logician in the third; it would seem that nobody in his right mind would long linger on the first.

Nevertheless, the *system of material implication* devised by Russell and Whitehead for *Principia Mathematica* is modeled after this first variety--their implication is so defined that it is always true when the consequent is true and is false only when the antecedent is true and the consequent false. It seems to this writer insufficiently emphasized in the book under discussion that when the aim of logic is construed as the derivation of true conclusions from true premises--when, in other words, we are being applied logicians--the use of material implication is eminently justified in that (1) it can be easily shown that we shall never get into trouble by using it--it is not possible to derive a false conclusion from true premises, and (2) implications are truth-functions of their constituent elementary propositions, thus amenable to truth-table checks. The "paradoxes of material implication" cease to be paradoxical when it is understood that what would be most unsatisfactory about an implication of the third kind might be entirely convivial in one of the first.

Lewis reminds us, however, that the aim of logic *can* be construed as the discovery and formulation of psychologically interesting tautologies; with respect to this quite distinct aim--that of pure logic--the language of *Principia*, though not false, contains much deadwood. Lewis's relation, strict implication, is defined contextually as follows: p strictly implies q if and only if both p and not q is logically impossible. With this definition--an intuitively and formally adequate explication of our third kind of implication, together with the definitions of material implication and equivalence, plus the few postulates peculiar to the *system of strict implication*, the entire *system of material implication* can then be deduced. Many but not all theorems are common to the two systems. An asserted strict implication can always be replaced by a material implication; the converse does not hold. When and only when a material implication is the major operation in a *tautology* can it be replaced by a strict implication. The *system of strict implication* is, thus, distinct from and both narrower and stronger than that of material implication; this fact, together with the notation of Lewis's system, render the laws of logic and their deduction--as evidenced in this important book--beautifully parsimonious.

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(continued on page 140)